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Sulla viscosità di volume - (I).

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Riassunto. — Secondo una teoria sviluppata da ECKART una conseguenza dell'esistenza di un secondo coefficiente di viscosità è la genesi di una corrente fluida in un mezzo attraversato da ultrasuoni, ed il LIEBERMANN effettivamente è riuscito a mettere in evidenza tale moto e a dare una misura di questo secondo coefficiente per alcuni liquidi. In questa prima nota noi riportiamo misure fatte per l'acqua in condizioni sperimentali poco diverse da quelle del LIEBERMANN ottenendo in linea di massima accordo con le previsioni teoriche; ma compaiono pure alcune particolarità, che richiedono ulteriori indagini sperimentali, quali, per esempio, una notevole variazione del secondo coefficiente per l'acqua per frequenze uguali a 1,8, 3,0 e 4,2 MHz.

L'esistenza per i liquidi di due coefficienti di viscosità è un'affermazione molto antica, ed è una logica deduzione dello sviluppo della teoria della viscosità, quando il problema viene trattato nella sua massima generalità. Esistono metodi diversi per l'introduzione di questi due coefficienti (quello della « shear viscosity » e quello della « bulk viscosity ») (1). Si è sempre ammesso, senza che sia data di questa affermazione una dimostrazione e una conferma sperimentale o teorica, che il coefficiente della « bulk viscosity » per i liquidi sia, come per i gas dipendente da quello della « shear viscosity », e che dunque praticamente non sia da esso distinto.

(1) C. SCHÄFFER: *Einf. in die Theor. Phys.*, Bd. 1, p. 917; *Handb. der Phys.*, Bd. 7 (L. HOPF: « Zähe Flüssigkeiten »), p. 93.

L'ECKART (2) partendo dalle equazioni di moto dei liquidi viscosi, nel caso che nel liquido si abbiano moti oscillatori con ampiezza non trascurabile, e cioè nel caso che nel liquido si propaghino ultrasuoni di una certa intensità, ha mostrato che trattando il problema con approssimazioni successive si giunge alla conclusione che in speciali condizioni geometriche, e cioè fascio di raggi ultrasonoro cilindrico che si propaga in un cilindro di diametro maggiore, limitato in lunghezza, senza la presenza di riflessione all'estremo del tubo, si genera un moto del liquido, moto che parte dalla sorgente, con una velocità v la quale, a parte altri parametri, è dipendente dai due coefficienti di viscosità. Dunque, realizzando un metodo sperimentale opportuno, che permetta la misura di questa velocità v, e sperimentando con vari liquidi, si può in primo luogo verificare nei suoi punti fondamentali la teoria sviluppata da ECKART e poi riscontrare sperimentalmente se effettivamente un liquido sia individuato da due coefficienti di viscosità.

Le misure sono state fatte da LIEBERMANN (3), e questo autore ritiene, in base alle sue misure, di poter affermare che effettivamente nei liquidi si può parlare di due coefficienti di viscosità, e per alcuni di questi liquidi questo autore dà anche i valori di tali coefficienti. Data l'importanza dell'argomento, e tenuto ancora conto del fatto che l'autore riporta molto sinteticamente le misure fatte, noi abbiamo ritenuto in primo luogo opportuno ripetere le misure, attenendoci più o meno allo stesso metodo. Ed in effetti può dirsi che l'unica modifica sostanziale che noi abbiamo attuata, in queste ricerche che qui esponiamo, consiste nel fatto che si è sostituito ad un sistema di misura delle forze, e quindi della velocità del liquido, basato su modifiche prodotte in un circuito elettrico, un metodo che permette una misura diretta del momento, e quindi della forza generata su di un disco dal moto del liquido attraversato da un fascio di ultrasuoni.

ECKART ritiene (ed in verità in maniera non molto convincente) che questa forza sia proporzionale al quadrato della velocità v; dunque, la misura delle forze ci dà la possibilità di risalire al valore della velocità acquistata dal liquido. Tale valore dipende, naturalmente, da vari parametri, alcuni dei quali geometrici, e dal secondo coefficiente di viscosità; e cioè indicando con r_0 il raggio del fascio ultrasonoro e con r quello del tubo, e ponendo:

$$G = \frac{1}{2}\left(\frac{r^2}{r_0^2} - 1\right) - \lg \frac{r}{r_0}, \tag{1}$$

indicando con la ϱ la densità del liquido, con c la velocità del suono in esso,

(2) C. ECKART: *Phys. Rev.*, **73**, 68 (1948); cfr. S. M. KARIM e L. ROSENHEAD: *Rev. of Mod. Phys.*, **24**, 108 (1952).

(3) L. N. LIEBERMANN: *Phys. Rev.*, **73**, 537 (1948); **75**, 1415 (1949).

con ω la pulsazione dell'onda ultracustica, con I l'intensità e con η' e η i due coefficienti di viscosità (nel caso dei gas $\eta' = -{}^2/_3\,\eta$) questa velocità v è data da:

$$v = G\,\frac{r^2\omega^2}{\varrho c}\cdot I\cdot\left(2+\frac{\eta'}{\eta}\right). \tag{2}$$

Liebermann nelle sue esperienze a proposito di tale problema, ha proceduto nel modo seguente: come risulta dalla formula (2) la velocità v è proporzionale a I, ed, in base a quanto è stato ammesso da Eckart, la forza prodotta dal flusso del liquido che è proporzionale a v^2, essa è dunque proporzionale a I^2; il liquido che è contenuto nel tubo di raggio r è limitato da una parete la quale teoricamente non dovrebbe essere riflettente, ma completamente trasparente, ciò non può realizzarsi in pratica; questa parete è dotata dunque di un potere riflettente R, e quindi quando si chiude il circuito ed il fascio ultrasonoro colpisce questa parete, compare una forza F_R che dipende dalla pressione di radiazione, e questa forza interviene immediatamente con un valore dato da:

$$F_R = \frac{2\pi r^2}{c}\,R\cdot I\,. \tag{3}$$

La forza dipendente dal moto del liquido è data invece da:

$$F_S = k\varrho v^2 \tag{4}$$

ed impiega un certo tempo per l'inerzia del liquido a raggiungere il suo valore. Dunque, studiando come variano nel tempo le forze applicate al disco, si deve sperimentalmente riscontrare una forza F_R proporzionale ad I, che compare istantaneamente alla chiusura del circuito degli ultrasuoni, ed un'altra F_S che richiede un certo tempo per raggiungere il valore finale, proporzionale a I^2.

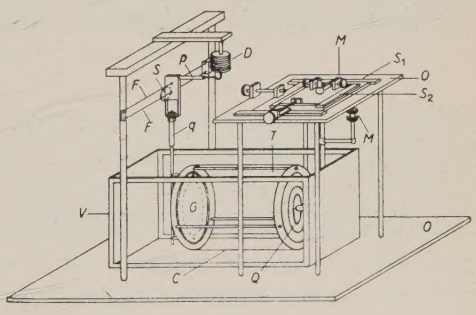

Fig. 1.

Il dispositivo sperimentale è rappresentato schematicamente in fig. 1. Il quarzo circolare Q di raggio r_0 è fissato sul fondo di un cilindro T di vetro di raggio r, sulla parte opposta è fissato un disco G che è portato da una leva q connessa ad una sospensione bifilare. Il tubo di vetro che circonda il quarzo mediante il dispositivo M può essere spostato in tutte le direzioni, ed inoltre il suo asse può essere orientato a piacere, in modo da ottenere (per tentativi) che il disco

sia situato proprio all'estremità del tubo, disposto normalmente all'asse di questo e dotato di completa mobilità.

In S è fissato uno specchietto, in modo che uno spostamento del disco G può tradursi in rotazione di un fascio luminoso. Questo fascio luminoso proviene da una fenditura F_1 (fig. 2), illuminata da una sorgente F ricca di raggi ultrarossi, ed alimentata da una batteria di accumulatori di grandissima capacità. L'immagine della fenditura F_1, prodotta dalla lente L_1, riflessa dallo specchietto S e proiettata in T da una lente L_2, è ridotta puntiforme dalla lente cilindrica L_3 e va a cadere sulla lamina bimetallica di una termopila di Moll, in un punto equidistante dalle due saldature S_1 e S_2. Una rotazione anche piccolissima dello specchietto S, e quindi uno spostamento di questa immagine, produce una differenza di temperatura nelle due saldature e quindi una corrente termoelettrica, accusata da un galvanometro di Moll, che viene poi registrata su carta sensibile avvolta su di un cilindro registratore. Il dispositivo bifilare (fig. 1) porta ancora un braccio p solidale alla leva q ed all'estremità di questo braccio si possono far cadere pesetti di valore noto mediante l'uso di un piccolo elettromagnete D; in tal modo sulle leve q si possono far agire forze, e quindi momenti noti, che permettono di fare la taratura in dine della deviazione al galvanometro sulla carta registratrice. In fig. 3 sono riportate alcune registrazioni di tali tarature.

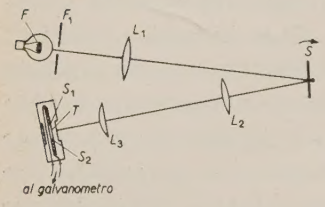

Fig. 2.

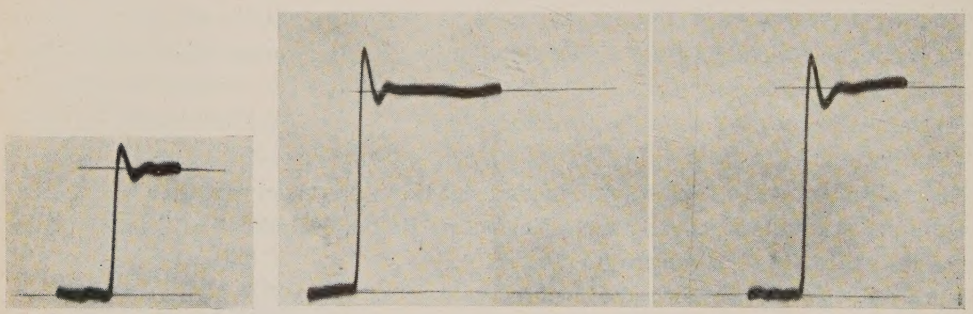

Fig. 3 *a*). Fig. 3 *b*).

Le misure venivano eseguite nel modo seguente: alla chiusura del circuito si misurava una deviazione D che mediante la taratura fatta poteva essere trasformata in dine F_R e queste dine F_R sono proporzionali all'intensità I; dopo un certo tempo si giungeva ad una deviazione finale D_2, e la differenza $D_2 - D_1$ era trasformata in dine F_S; queste dine F_S devono essere proporzio-

nali a v^2 e quindi a I^2; riportando come ascisse F_R e come ordinate $\sqrt{F_S}$ i punti figurativi devono essere allineati su di una retta.

Il coefficiente angolare di questa retta da (2), (3) e (4) risulta:

$$\frac{\sqrt{F_S}}{F_R} = \frac{\sqrt{k}\,\omega^2}{2\pi R}\,\frac{G}{\sqrt{\varrho}\,c^3}\left(2+\frac{\eta'}{\eta}\right); \tag{5}$$

esso dipende da parecchi parametri; fra cui, come vedesi dalla formula, anche da un parametro $(2+\eta'/\eta)$ dipendente dal secondo coefficiente. Bisogna però ricordare che queste misure sono basate sulla valutazione di I mediante la misura della pressione di radiazione, la quale è misurabile solo se la parete è in parte riflettente, mentre per la teoria di Eckart è necessario e fondamentale che la parete sia assolutamente trasparente. Dunque si misura in condizioni che non corrispondono perfettamente a quelle richieste dalla teoria.

Il liquido adoperato in queste prime ricerche è l'acqua, ed è posta nella vaschetta V (fig. 1), in modo da riempire completamente il cilindro T. Il quarzo era eccitato mediante un oscillatore sulla frequenza di 1,8 MHz (terza armonica della fondamentale); esso aveva diametro $2r_0$ di 3 cm, ed il tubo T un diametro $2r$ di 6 cm. Il disco tenuto da una ghiera era costituito da una lamina di Al dello spessore di $1{,}20\cdot10^{-3}$ cm ed esperienze preliminari hanno dimostrato che esso poteva considerarsi trasparente agli ultrasuoni, poichè R ha per la frequenza di 1,8 MHz un valore di 0,014. Posta la sorgente in funzione, si ha al galvanometro una brusca deviazione iniziale dipendente dalla pressione di radiazione, che naturalmente compare all'istante; successivamente la deviazione aumenta, e raggiunge il suo massimo valore definitivo con una certa lentezza. Si è visto, sperimentalmente, usando anche altri liquidi, che il tempo, necessario a che la deviazione raggiunga il massimo, a parità di condizioni sperimentali, dipende dal liquido adoperato, ed a parità di liquido tale tempo, inoltre, diminuisce con la frequenza degli ultrasuoni usati.

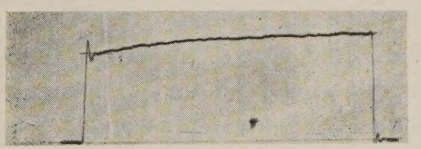

Fig. 4.

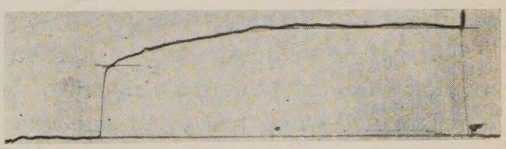

Fig. 5.

In fig. 4 è riportata una delle registrazioni ottenute con un fascio di ultrasuoni di frequenza $\nu = 1{,}8$ MHz.

Nella fig. 5 è invece riportata una registrazione relativa all'acqua, con la frequenza di oscillazione $\nu = 3{,}0$ MHz.

La misura di D_1, e quindi, conseguentemente, la distinzione tra le due forze

F_S e F_R è tanto più precisa quanto più piccolo è il periodo di oscillazione del pendolo, e cioè quanto più pronto è l'apparecchio. In base alla relazione (5), e indicando con $h = \sqrt{k}/2\pi$ un coefficiente di proporzionalità che non conosciamo si ha:

$$\frac{\sqrt{F_S}}{F_R} = h \frac{G}{R\sqrt{\varrho}} \frac{\omega^2}{c^3}\left(2 + \frac{\eta'}{\eta}\right). \tag{6}$$

E cioè a parità di liquido, il rapporto scritto deve essere costante qualunque sia il valore di I, se resta costante la pulsazione ω ed il parametro G.

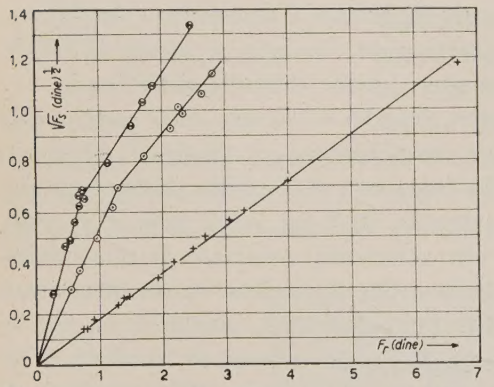

Fig. 6. – + $\nu = 1{,}8$ MHz; ⊙ $\nu = 3{,}0$ MHz; ⊗ $\nu = 4{,}2$ MHz.

Nella fig. 6 si riportano i risultati ottenuti per l'acqua con il disco di 60 mm (raggio del fascio ultrasonico $r_0 = 1{,}5$; $G = 0{,}553$). Sull'asse delle ascisse sono riportati direttamente i valori di F_R espressi in dine e su quello delle ordinate i valori $\sqrt{F_S}$ in (dine)$^{\frac{1}{2}}$. Le misure sono state condotte relativamente a tre frequenze e le tre curve sono appunto relative alle frequenze $\nu = 1{,}8$ MHz, 3,0 MHz e 4,2 MHz. Si può concludere come primo fatto fondamentale che effettivamente i punti si trovano su di una retta, secondo quanto è previsto dalla teoria di Eckart. Ora, in queste misure, come anche in una curva riportata nel lavoro di LIEBERMANN, si osserva un fatto caratteristico il cui significato è ancora oscuro: che i punti rappresentativi relativamente alle frequenze più elevate: $\nu = 3{,}0$ e $\nu = 4{,}2$ MHz cadono su due rette con due coefficienti angolari diversi, uno per le basse ed uno per le grandi intensità sonore.

La ragione di questo fatto è attribuita dal LIEBERMANN, ma senza dimostrazione, ad una differenza di moto del liquido, e cioè ad uno scorrimento lamellare nel primo tratto, ed uno scorrimento turbolento nel secondo.

Ci riserviamo, in un lavoro successivo, di ritornare particolarmente su tale importante questione.

La teoria fa prevedere inoltre una proporzionalità del coefficiente di inclinazione della retta rispetto a G. Ora, noi abbiamo fatto anche misure con tubo di raggio $r = 4{,}5$ cm, in tal modo abbiamo mutato il valore di G, e precisamente siamo passati dal valore $G = 0{,}553$ al valore $G' = 1{,}00$; dunque $G/G' = 0{,}553$.

I punti figurativi relativi a questa nuova serie di esperienze sono riportati

in fig. 7, nella quale è anche tracciata la retta ottenuta alla stessa frequenza, col tubo di raggio $r = 3{,}0$ cm. Tali punti cadono su di una retta il cui coefficiente angolare è diverso da quello precedente, ed il rapporto tra questi due coefficienti vale 0,43; se si tien conto delle difficoltà che si ritrovano nella misura, si può dire che tale numero è in accordo accettabile col valore del rapporto G/G'.

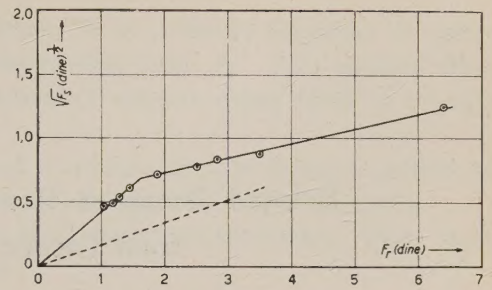

Fig. 7. – - - - - H_2O Disco $\varnothing = 60$ mm, $\nu = 1{,}8$ MHz; ⊙ H_2O Disco $\varnothing = 90$ mm, $\nu = 1{,}8$ MHz.

Dalla relazione (6) risulta ancora che a parità di liquido il rapporto $\sqrt{F_s/F_R}$ è proporzionale a ω^2/R, essendo R il potere riflettente; tale potere riflettente dipende anch'esso dalla frequenza secondo la relazione:

$$(7) \qquad R = \frac{\left[\frac{\varrho_1 c_1}{\varrho_2 c_2} - \frac{\varrho_2 c_2}{\varrho_1 c_1}\right]^2}{4 \operatorname{cotg}^2 2\frac{\pi s}{\lambda} + \left[\frac{\varrho_1 c_1}{\varrho_2 c_2} + \frac{\varrho_2 c_2}{\varrho_1 c_1}\right]^2},$$

dove s e $\lambda = c/\nu$ sono rispettivamente lo spessore della lamina e la lunghezza d'onda degli ultrasuoni nella lamina d'Al, ϱ_1, c_1 e ϱ_2, c_2 rispettivamente la densità e la velocità degli ultrasuoni nel liquido e nella lamina.

In fig. 6, come si è detto, sono riportati i valori sperimentali ottenuti relativamente alle tre frequenze $\nu_1 = 1{,}8$ MHz, $\nu_2 = 3$ MHz, $\nu_3 = 4{,}2$ MHz. I coefficienti angolari delle curve riportate relativamente al primo tratto, e cioè i rapporti $\sqrt{F_s/F_R}$, stanno tra loro come 1 : 2,96 : 5,08, mentre i quadrati delle ω divisi per i rispettivi R stanno tra loro come 1,00 : 1,02 : 1,06; contrariamente quindi a quanto previsto dalla (6), in base a queste prime misure, nel caso dell'acqua, il coefficiente angolare non è proporzionale a ω^2/R, e ciò può forse attribuirsi ad una variazione di η'/η con la frequenza.

Anche tale importante punto sarà oggetto di ulteriori studi.

SUMMARY

According to a theory developed by ECKART, a consequence of the existence of a second viscosity coefficient is the genesis of a fluid current in a medium crossed by ultrasonic waves; LIEBERMANN has really succeded in putting in evidence this motion and giving for some liquids a measure of this second coefficient. In this first note we are giving some measurements made for water in experimental conditions nearly similar to those of LIEBERMANN, obtaining results accordingly to the theoretical previsions; but there are also some peculiarities which require further experimental investigation; for istance the considerable variation of the second coefficient for water at frequencies of 1,8, 3,0 and 4,2 MHz.

Delayed Decay of Heavy Fragments Ejected from Cosmic Ray Stars.

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(ricevuto il 16 Febbraio 1954)

Summary. — Data concerning seven cases of production and decay of unstable nuclear fragments are compared and different interpretations discussed. In all the seven cases the unstable fragments are produced in interactions the energy of which is in the range from a few to some tens of GeV and which correspond to more or less central collisions of a neutral or singly charged particle with a heavy nucleus in the emulsion. The life-time of the unstable fragments is longer than 10^{-12} s. The evaluation of the total energy released in their decay is consistent with a unique Q-value of 175 MeV, but not with the value of 140 MeV. Energy distribution among the charged secondaries shows a marked non-uniformity. In the majority of cases no mesons are observed to be emitted among these secondaries. Considering the two modes of decay of a V_1^0-particle within the nucleus: the mesonic and non-mesonic decay ([6]), all the discussed features of the seven cases support the bound V_1^0-hypothesis.

In 1952 Danysz and Pniewski ([1]) reported a case of a peculiar coincidence of two events recorded in nuclear emulsion. They ascribed it tentatively to a delayed decay of a heavy fragment ejected from a cosmic ray star. Soon afterwards similar observations have been made in other laboratories ([2-5]). Recently we have found two more cases of this type. In all these cases a

([1]) M. Danysz and J. Pniewski: *Bull. de l'Ac. Polon. des Sci.*, **1**, 42 (1953); see also *Phil. Mag.*, **44**, 348 (1953).

([2]) D. A. Tidman, G. Davis, A. J. Herz and R. M. Tennent: *Phil. Mag.*, **44** 350 (1953).

([3]) M. M. J. Crussard et D. Morellet: *Compt. Rend.*, **236**, 64 (1953).

([4]) G. Lovera, L. Barbanti Silva, C. Bonacini, C. de Pietri, R. Perilli Fedeli and A. Roveri: *Nuovo Cimento*, **10**, 986 (1953).

([5]) P. S. Freier, G. W. Anderson and J. E. Naugle: private communication.

multiply charged particle (further referred to as a heavy fragment or particle *f*) is ejected from a high energy interaction (star *A*). This particle after having apparently stopped in the emulsion (*) decays giving birth to a small low energy star (star *B*).

In view of the increased amount of available data, it seems a matter of interest to see how they fit the different explanations under discussion.

Let us consider the question as put by DANYSZ and PNIEWSKI. They have discussed five possible interpretations:

1) chance coincidence,
2) collision with an emulsion nucleus,
3) excited nucleus,
4) mesonic-ion,
5) V_1^0-particle bound in a nucleus.

The cumulated evidence of all the seven observations rules out the first three possibilities and establishes the existence of some new phenomenon or phenomena.

The plausibility of the mesonic-ion hypothesis depends both on the probability of a π-meson being picked up by the fragment *f* on one of its quantum orbits (star *A*), and on the probability of a further absorption of such a meson by the nucleus *f* (star *B*). According to CHESTON and PRIMAKOFF (6) the life-time of a mesonic-ion of this kind might be long enough to account for the experimental data, however, the probability of the pick-up process seems to us rather small. The average kinetic energy of π-mesons ejected from a high energy primary interaction is much larger than the binding energy of a π-meson in the mesonic-ion. A direct test of this hypothesis might be provided by the evaluation of the total energy release in the decay process: $Q =$ or $\neq 140$ MeV.

The hypothesis of the bound V_1^0 requires the life-time of the *f* fragment to be rather shorter than the life-time of free V_1^0-particles ($3 \cdot 10^{-10}$ s) and the total energy release not to be significantly different from 175 MeV (including the rest-mass of π-meson if emitted).

According to CHESTON and PRIMAKOFF a V_1^0-particle bound within a nucleus may disintegrate either by normal (mesonic) decay:

$$V_1^0 \to p + \pi^- \quad \text{and} \quad V_1^0 \to n + \pi^0 \quad (?)$$

or by interaction with other nucleons of the nucleus, changing into a nucleon

(*) Perhaps with the exception of event 4 and 7 (Table I).

(6) W. B. CHESTON and H. PRIMAKOFF: *International Congress on Cosmic Radiation*, Bagnères-de-Bigorre (1953).

with 175 MeV energy release: the non-mesonic decay. CHESTON and PRIMAKOFF have calculated the branching ratio of non-mesonic to mesonic decay, assuming that V_1^0-particles are produced copiously in pairs. They obtained the value of 5 to 50 increasing with the number of nucleons in the nucleus.

Hence, for the V-nuclei (nuclei with a bound V_1^0-particle) we should expect:

1) that in the majority of cases they would decay without a π-meson emission,

2) that both their life-time and the relative probability of their mesonic to non-mesonic decay would increase with the decrease of their mass number.

Moreover if we consider the non-mesonic decay of V_1^0-particles as a result of a two body interaction:

$$V_1^0 + p \text{ (or n)} \rightarrow n + p \text{ (or n)} + 175 \text{ MeV},$$

we should expect to observe as charged decay products of the V-nuclei only very low energy particles with at most one more energetic proton among them. On the average the kinetic energy of this proton would be of the order of 80 MeV largely spread around this value due to the internal motion of the nucleons in the nucleus.

Let us now look at the experimental evidence available.

These data are summarized in Table I, while Table II gives the directions of tracks in star *B* for the events 1, 6 and 7.

The fact that till now in different laboratories all over the world as many as seven events of the type discussed have been found makes the probability of a chance coincidence for all of them extremely small. Indeed such a probability for one event only is evaluated to be of the order of 10^{-4} [2].

In five cases out of the seven the fragment *f* is apparently stopped in the emulsion. Its kinetic energy (if any) must be certainly much lower than the observed energy released in the stars *B*, and so the possibility of collision with an emulsion nucleus must be discarded. In the remaining two cases (event 4 and 7) the situation is not so clear due to the shortness of the tracks *f*.

The time of flight of the fragments *f* being longer than 10^{-12} s yields a lower limit for their life-time $\tau > 10^{-12}$ s, a time much too long for the life of a highly excited nucleus.

The energies of the primary interactions range from a few to some tens of GeV. The large number of heavy secondaries indicates that stars *A* are due to more or less central collisions.

In all the seven cases the small low energy stars (stars *B*) may correspond to the decay of the fragment *f* with a unique *Q*-value. However in the majo-

TABLE I.

Event			1	2	3	4	5	6	7
Type of star A			$21+18p$	$16+0p$	$30+30p$	$13+1p$	$24+10p$	$18+14p$	$18+1p$
Track f	lenght (μ)		90	219	68	2	30	80	13
	charge (e)		4–6	2–3	2–3	—	4–7	3–4	—
Star B	number of tracks		4	3	3	4	3	3	2
	range (μ) and grain density of track:	1	10 black	24 black	1,4 recoil	2245 black	5,8 black	14 black	113 black
		2	126 black	117 black	60 black	>192 black	182 black	78 black	>330 95 g/50 μ $R=4$ mm*
		3	>674 55 g/50 μ $R=23$ mm*	5,1 g_{pl}	>325 2,3 g_{pl}	169 black	>759 2,56 g_{pl}	>370 80 g/50 μ $R=6,5$ mm*	—
		4	2,5 recoil	—	—	>262 black	—	—	—
Authors			DANYSZ and PNIEWSKI	TIDMAN *et al.*	CRUSSARD and MORELLET	LOVERA *et al.*	FREIER *et al.*	present work	present work

* if proton.

TABLE II.

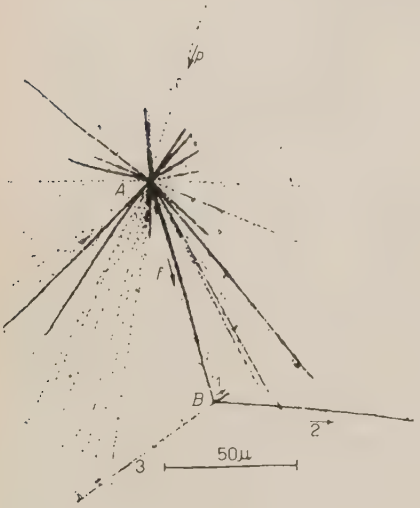
Event	1 (Pl. 1)				6 (Pl. 2)			7 (Pl. 3)	
Track	1	2	3	4	1	2	3	1	2
φ	0°	22°	162°	135° to 200°	0°	178°	285°	0°	137°
Θ	— 49°	+ 34°	— 36°	0° to 90°	— 47°	+ 12°	+ 48°	— 29°	+ 49°

φ - angle in plane of the emulsion.
Θ - angle of dip.

rity of cases the evaluation of these Q-values depends essentially on the mass of the fastest charged decay particle. In what follows and whenever it is possible we shall tentatively suppose that these particles are protons.

Event 1 (Pl. 1). – The charges of the secondaries (star B) as evaluated by inspection are:

$$Z_1 = 1e \quad \text{or} \quad 2e$$
$$Z_2 = 1e \quad \text{or} \quad 2e$$
$$Z_3 = 1e$$
$$Z_4 = \text{undetermined}.$$



Pl. 1.

The mass of particle 3, due to the shortness of its track in the emulsion is undetermined, scattering and grain-density measurements suggest here rather a protonic mass. However, following the suggestions made by Dr. CRUSSARD (*) we have analysed this case assuming two alternative decay schemes:

$$^{11}B_5^* \text{ (V-Boron)} \rightarrow {}^4He_2 + {}^1H_1 + \pi^- + {}^6Li_3 ,$$
$$^{11}B_5^* \text{ (V-Boron)} \rightarrow {}^1H_1 + {}^1H_1 + \pi^- + {}^9Be_4 .$$

In both cases the momenta of the secondary particles were balanced within large limits of error (mainly due to the poorly determined momentum of the short recoil). A calculation was made of the Q-value of the assumed mesonic decay of the V_1^0-particle taking for the V_1^0-particle the same binding energy in the parent nucleus as for the neutron. The calculation yields: 37 and 35 MeV for the first and second scheme respectively. The agreement with the generally accepted Q-value of the V_1^0-particle decay is excellent. We are inclined, however, to consider it rather as accidental.

Further, star B was analysed assuming particle 3 to be a proton. The Q-values were calculated for different decay schemes corresponding to different charges and masses of the f-fragment as of the charged decay products. In these decay schemes where two or three neutrons were assumed to be emitted, a lower limit of the Q-value was determined together with the Q-value corresponding to an equal amount of energy taken away by protons as by neutrons ($\bar{Q}$). In two instances only this lower limit dropped below 140 MeV (132 MeV and 139 MeV), the value of $\bar{Q}$ ranged from 190 to 230 MeV, whereas

(*) Private comunication.

in three decay schemes, where only one neutron was assumed to be emitted the values of the Q where 145, 170 and 175 MeV.

Event 2. – The charges of the secondaries as given by TIDMAN *et al.* are:

$$Z_1 = 1e \quad \text{or} \quad 2e$$

$$Z_2 = 1e \quad \text{or} \quad 2e$$

$$Z_3 = 1e\,.$$

Scattering and grain-density measurements on track 3 suggest that this track is more likely to be that of a proton than that of a π-meson (TIDMAN *et al.*). The momenta of charged secondaries cannot be balanced (Pl. 14 and 15, TIDMAN *et al.*).

Assuming particle 3 to be a π-meson, rough calculations based on published data yield for the Q of the mesonic decay of the V_1^0-particle inside the nucleus values significantly higher than 35 MeV whereas assuming it to be a proton, the calculations of the $\overline{Q}$ yields respectively 140 MeV and 150 MeV for ^{6}Li and ^{7}Li taken as f-particles.

Event 3. – In this case a light particle of mesonic mass is observed among the secondaries of star B.

According to CRUSSARD and MORELLET it seems preferable to view this event as an example of a mesonic decay of a V-Helium or V-Lithium nucleus. The two reactions considered by these authors are:

$$^4\mathrm{He}_2^* \text{ (V-Helium)} \rightarrow {}^3\mathrm{He}_2 + {}^1\mathrm{H}_1 + \pi^-\,,$$

$$^7\mathrm{Li}_3^* \text{ (V-Lithium)} \rightarrow {}^6\mathrm{Li}_3 + {}^1\mathrm{H}_1 + \pi^-\,.$$

In both cases the momenta of the secondaries of star B are balanced, and the Q-values of the assumed V_1^0-particle decay are: 48 ± 4 MeV and 35 ± 4 MeV respectively (CRUSSARD and MORELLET).

Event 4. – Due to the shortness of the track interconnecting the two stars A and B, it is difficult to draw any definite conclusion. The energy release observed in star B is evaluated to be of the order of 100 MeV (LOVERA *et al.*).

Event 5. – In the case reported by FREIER *et al.* the tracks of the three secondaries are coplanar within large limits of error due to the shortness of track 1 ($4^\circ \pm 5^\circ$). Tracks 2 and 3 correspond to singly charged particles.

Scattering and grain-density measurements do not allow to decide whether particle 3 is a proton or a meson.

The most likely scheme as discussed by FREIER *et al.* is that of a non-mesonic decay of a V_1^0 within a V-Nitrogen nucleus:

$$^{14}N_7^* \text{ (V-Nitrogen)} \rightarrow {}^1H_1 + {}^1H_1 + {}^{12}B_5 ,$$

with a Q-value of 168 ± 11 MeV. Although the proposed decay scheme agrees well with all experimental data, the authors do not exclude other interpretations of the decay process.

Pl. 2.

Event 6 (Pl. 2). – The charges of the secondaries as evaluated by inspection are:

$$Z_1 = 1e \quad \text{or} \quad 2e$$

$$Z_2 = 1e \quad \text{or} \quad 2e$$

$$Z_3 = 1e .$$

Track 3 has been followed in the facing plate for another 800 μ (total observed length being 1 200 μ).

The possibility that particle 3 is a meson is ruled out by grain-density and scattering measurements which indicate that it is a proton rather than a deuteron.

Assuming particle 3 to be a proton, the Q, Q-limits and $\overline{Q}$-values for different possible decay schemes were calculated. The lower limit of Q ranges from 80 to 100 MeV, the Q-values for one neutron decay schemes, from 80 to 120 MeV, while the $\overline{Q}$-values, from 110 to 130 MeV

Event 7 (Pl. 3). – The charges of the secondaries and of the *f*-fragment as evaluated by inspection are:

$$Z_1 = 1e \quad \text{or} \quad 2e \quad \text{(more likely } 1e\text{)}$$

$$Z_2 = 1e$$

$$Z_f = \text{undetermined} .$$

While the general appearance of the *f*-particle track corresponds rather to an end part of a track, no definite conclusion can be drawn as to its charge.

Track 2 has been followed in the facing plate for another 800 μ so the possibility of particle 2 being a meson was ruled out. Both grain-density and scattering measurements are consistent with the assumption that it is a proton.

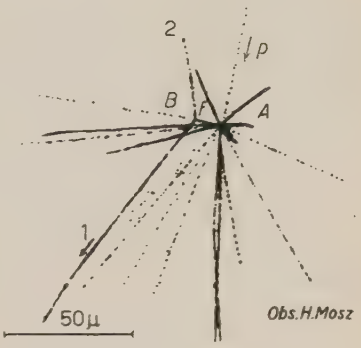
Due to the shortness of track *f* it is difficult to draw any more definite conclusions here.

Assuming particle 2 to be a proton the $\bar{Q}$-value would be of the order of 100 MeV.

Summarizing we should conclude that:

1) *f*-fragments are observed to be emitted from interactions of not too high energy (from a few to some tens of GeV).

2) All these interactions correspond to rather central collisions of neutral or singly charged particles with heavy nuclei of the emulsion.



Pl. 3.

3) The experimental evidence does not exclude a unique *Q*-value of 175 MeV for all the observed *f*-fragment decays. In two cases (event 3 and 5) the *Q*-value is certainly above 140 MeV and probably above that in the remaining cases.

4) The emission of a light meson among the secondaries of the *f*-fragment decays has been definitely observed in one case (event 3). It certainly does not take place in event 4, 6 and 7, and is rather improbable in the remaining cases.

5) The energies of the secondaries as listed in Table III in the majority of cases show a marked non-uniformity: a faster particle among comparatively low energy ones.

TABLE III.

Event	1	2	3	4	5	6	7
Energy of secondaries (if protons) in MeV	82	48	25 (1)	23,6	125	40	30
	4,1	4,0	2,7	> 6,5	10,4 (2)	3,1	3,8
	0,8	1,6	recoil	> 5,4	5,2	1,0	—
	recoil	—	—	5,0	—	—	—

(1) if π
(2) if B^{12}

Although all the features discussed of the production, life and decay of the *f*-fragments seem to give support to the original bound V hypothesis, they do not rule out the possibility of other explanations, involving, as pointed out by FREIER *et al.*, other unstable particles or other decay processes.

Looking for further tests of this hypothesis it might be of interest to consider not only cases of decay at rest, but also cases of decay in flight. Assuming the existence of V-nuclei which are sometimes ejected from cosmic ray stars, one might expect to find examples of their decay in flight. The emulsion technique would offer here rather better conditions for the heavier V-nuclei decay in flight detection, while the cloud chamber seems to be more suitable for the detection of the V-deuteron decays. Such observations would offer additional information in regard to the evaluation of the life-time of these unstable particles.

We are grateful to Dr. P. S. FREIER *et al.* for sending us their paper before publication.

RIASSUNTO (*)

Si confrontano fra di loro e si discutono differenti interpretazioni dei dati concernenti sette casi di produzione e decadimento di frammenti nucleari instabili. In tutti i sette casi i frammenti instabili sono prodotti in interazioni la cui energia va da alcuni a alcune decine di GeV e corrispondenti a urti più o meno centrali di una particella neutra o dotata di carica unitaria con un nucleo pesante dell'emulsione. La vita media dei frammenti instabili supera 10^{-12} s. La valutazione dell'energia totale liberata si accorda con un unico valore per Q di 175 MeV ma non col valore di 140 MeV. La distribuzione dell'energia fra i secondari carichi mostra una spiccata disuniformità. Nella maggioranza dei casi, fra questi secondari non si osservano mesoni. Considerando i due modi di decadimento di una particella V_1^0 entro il nucleo, il decadimento mesonico e non mesonico ([6]), tutte le caratteristiche riportate pei sette casi sono a sostegno dell'ipotesi della V_1^0 legata.

(*) *Traduzione a cura della Redazione.*

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Sulla indipendenza dalla carica nella produzione di particelle Λ.

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(ricevuto il 22 Febbraio 1954)

Riassunto. — Si discutono alcune conseguenze dell'ipotesi dell'indipendenza dalla carica applicata a processi semplici di produzione di particelle Λ, di taluni dei quali si è avuta recentemente indicazione sperimentale; per completezza vengono anche riportate altre conseguenze della ipotesi della indipendenza dalla carica ove la si ritenga applicabile anche ai processi di decadimento. Se si assegna uno spin isotopico semintero alle particelle Λ, ad ogni processo di produzione di Λ^0 ovviamente corrisponde un processo di produzione di Λ^+ avente la stessa sezione d'urto. Se si fa l'ipotesi della indipendenza dalla carica è possibile verificare che tutte le sezioni d'urto di produzione di Λ^0 (e di Λ^+) possono venire maggiorate, ed in maniera molto restrittiva, con sezioni d'urto di produzione di Λ^{++} (o, se si preferisce di Λ^-). Se si assegna uno spin isotopico intero alle particelle Λ (schema di Gell-Mann) si può ancora verificare che tutte le sezioni d'urto di produzione di Λ^0 possono venire maggiorate con sezioni d'urto di produzione di Λ^- (o, se si preferisce, di Λ^+). D'altra parte lo schema di Gell-Mann, con l'aggiunta di altre ragionevoli ipotesi, implica, in qualsiasi processo di produzione, la produzione di un uguale numero (per « numero » intendendo la differenza tra il numero delle particelle ed il numero delle antiparticelle) di nuovi fermioni e di nuovi bosoni, e questa conseguenza non dovrebbe essere lontana da una verifica sperimentale. Può darsi che le precedenti conclusioni difficilmente risultino conciliabili con l'attuale apparenza sperimentale di una molto maggiore abbondanza di particelle Λ neutre rispetto a particelle Λ cariche (ci riferiamo in particolare alle particelle Λ^{++} e Λ^-).

Si ritiene generalmente che nelle interazioni tra nucleoni (almeno fino ad energie non molto alte) e nelle interazioni tra nucleoni e pioni l'indipendenza dalla carica (invarianza per rotazioni nello spazio dello spin isotopico) sia rispettata; recentemente si sono poi avute conferme sperimentali sufficien-

temente attendibili di questo fatto ([1,1']). Non è ancora noto se l'indipendenza dalla carica sia rispettata in processi cui prendono parte le nuove particelle instabili da poco scoperte: se, cioè, è possibile assegnare a talune almeno di queste particelle uno spin isotopico, in maniera che la gran parte dei processi in cui esse intervengono possa venire spiegata mediante interazioni che rispettano l'indipendenza della carica. Due possibili processi di decadimento in tre pioni sono a priori possibili per il mesone τ positivo, che rispettano la conservazione della carica: il decadimento in tre pioni carichi, ed il decadimento in un pione carico e due pioni neutri. È noto che del primo processo di decadimento si ha ormai una chiara evidenza sperimentale, mentre solo pochi eventi finora si conoscono che si prestino ad essere interpretati come processi di decadimento del secondo tipo. Può darsi che le interazioni (forti) responsabili per la produzione siano di un carattere diverso delle interazioni (deboli) responsabili per il decadimento, in modo che le prime rispettano la indipendenza dalla carica, le seconde invece non la rispettano. È stato comunque osservato da DALITZ ([2]), che l'indipendenza dalla carica, applicata al processo di decadimento del mesone τ positivo in tre pioni, in effetti richiede solamente che la frequenza del decadimento in un pione carico e due pioni neutri non sia inferiore a un quarto della frequenza del decadimento in tre pioni carichi. Ove ancora si tenga conto della maggiore difficoltà sperimentale di rivelare il decadimento in un solo pione carico, si deve concludere che non è possibile, sulla base degli attuali dati, decidere della validità o meno della ipotesi della indipendenza dalla carica in questo particolare processo. In questa nota verranno discusse alcune conseguenze della ipotesi della indipendenza dalla carica applicata a processi semplici di produzione di particelle Λ, di taluni dei quali si è avuta recentemente indicazione sperimentale ([3]) [definiamo le particelle Λ, che appartengono alla famiglia degli iperoni, Y, dal loro processo di decadimento $\Lambda \rightarrow$ nucleone + pione; inoltre, definiamo le particelle θ, che appartengono alla famiglia dei mesoni pesanti, K, dal loro processo di decadimento $\theta \rightarrow$ pione + pione ([4])]; verranno inoltre discusse altre conseguenze della ipotesi della indipendenza dalla carica, che si riferiscono ad altri processi, ma che sono collegate alle precedenti. Se si assegna uno spin isotopico semintero alle particelle Λ, Λ^0 e Λ^+ risultano stati simmetrici rispetto alla carica, ed, ovviamente, ad

([1]) Tra l'altro: E. FERMI: *Proceedings of the Third Annual Rochester Conference*, 1952.

([1']) R. H. HILDEBRAND: *Phys. Rev.*, **89**, 1090 (1953) insieme al precedente lavoro di M. N. WHITEHEAD e C. RICHMAN: *Phys. Rev.*, **83**, 855 (1951).

([2]) R. H. DALITZ: *Proceedings of the Phys. Soc.*, **66**, 710 (1953). Vedi anche A. GAMBA: *Nuovo Cimento*, **11**, 323 (1954).

([3]) W. B. FOWLER, R. P. SHUTT, A. M. THORNDIKE e W. L. WHITTMORE, non ancora pubblicato (nel seguito questo lavoro verrà indicato con F.S.T.W.).

([4]) Vedi *Nuovo Cimento*, **11**, 213 (1954).

ogni processo di produzione di Λ^0 corrisponde un processo di produzione di Λ^+ avente la stessa sezione d'urto, se solo si fa la ipotesi della simmetria rispetto alla carica. Se ancora si fa l'ipotesi della indipendenza dalla carica è possibile verificare che tutte le sezioni d'urto dei processi di produzione di Λ^0 e Λ^+ possono venire maggiorate, ed in maniera molto restrittiva, con sezioni d'urto di processi di produzione di Λ^{++} (o, se si preferisce, con sezioni d'urto di processi di produzione di Λ^-). In particolare, se si vuole che le sezioni d'urto per produzione di Λ^{++} (o di Λ^-) si annullino, le altre sezioni d'urto devono tutte parimenti annullarsi. Se si assegna uno spin isotopico intero alle particelle Λ (schema di Gell-Mann) Λ^0 e Λ^+ non sono più stati simmetrici rispetto alla carica, si può però ancora verificare che tutte le sezioni d'urto di produzione di Λ^0 possono venire maggiorate con sezioni d'urto di processi di produzione di Λ^- (o, se si preferisce, con sezioni d'urto di processi di produzione di Λ^+). In particolare, se si vuole che le sezioni d'urto dei processi di produzione di Λ^- (o di Λ^+) si annullino, anche le sezioni d'urto dei processi di produzione di Λ^0 devono tutte parimenti annullarsi. Riguardo allo schema di Gell-Mann, si può dire che, in generale, comunque complicato sia il processo, purchè però si conservi almeno la componente z dello spin isotopico totale e valga inoltre la conservazione delle particelle pesanti, la produzione deve sempre avvenire in coppie di un nuovo fermione più un nuovo bosone; questa conseguenza della ipotesi di Gell-Mann non dovrebbe essere lontana da un controllo sperimentale.

Due casi del processo di produzione pione+nucleone→mesone pesante+ +particella Λ, sono stati recentemente osservati in camera a nebbia da Fowler, Shutt, Thorndike e Whittmore ([3]). Il meccanismo di produzione a coppie risulta necessario nella teoria di Pais ([5]): tuttavia altre ipotesi possono ancora condurre ad un meccanismo di produzione a coppie ([6]), ed in ogni caso non è chiaro se veramente sia necessario aggiungere nuove regole alla indipendenza dalla carica, che pertanto si intenderà qui formulata al modo solito e senza alcuna regola aggiuntiva.

Si consideri il processo: pione + nucleone → mesone pesante (θ) + particella Λ. Sia inoltre,

caso α: T_θ (spin isotopico del mesone pesante),= 0, T_Λ (spin isotopico della particella Λ) = 1/2; in questo caso esistono θ^0, Λ^+ e Λ^0 (se si considera l'evento E di F.S.T.W. questa assegnazione di spin isotopici è sbagliata; naturalmente questo stesso vale anche se solo si considera l'esistenza, quale già si sa da altre esperienze, di Λ^+, supposti, a parte la carica, identici ai Λ^0 e supposto che non siano antiparticelle, cosa del resto improbabile soprattutto per motivi energetici). Interviene un solo elemento di matrice, relativo allo

([5]) A. Pais: *Proceedings of the Lorenzt-Kammerling Onnes Conference.*

([6]) A. Pais: *Phys. Rev.*, **86**, 663 (1952); M. Gell-Mann: *Phys. Rev.*, **92**, 833 (1953).

stato con spin isotopico totale 1/2, pertanto una sola sezione d'urto è indipendente e tutte le altre sono ad essa collegate tramite, ad esempio, la relazione (od altra equivalente)

$$\sigma_\alpha(\pi^-\mathrm{P}\,|\,\theta^0\Lambda^0) = 2\sigma_\alpha(\pi^0\mathrm{P}\,|\,\theta^0\Lambda^+)\ ; \tag{1}$$

($\sigma(A\,|\,B)$ è la sezione d'urto differenziale del processo $A\to B$). La precedente relazione è l'unica condizione che la indipendenza dalla carica pone alle sezioni d'urto una volta che siano soddisfatte le altre relazioni indipendenti (due in questo caso) che esprimono la più debole condizione di simmetria rispetto alla carica: relazioni di quest'ultimo tipo per brevità non verranno riportate perchè possono costruirsi immediatamente.

Caso β: $T_\theta = 0$, $T_\Lambda = 3/2$, esistono θ^0, Λ^{++}, Λ^+ Λ^0, Λ^- (anche in questo caso non c'è posto per l'evento E di F.S.T.W., possono però esistere Λ^-). Interviene un solo elemento di matrice, relativo allo stato con spin isotopico totale 3/2, pertanto una sola sezione d'urto è indipendente e tutte le altre sono ad essa collegate tramite, ad esempio, la relazione

$$6\sigma_\beta(\pi^-\mathrm{P}\,|\,\theta^0\Lambda^0) = 3\sigma_\beta(\pi^0\mathrm{P}\,|\,\theta^0\Lambda^+) = 2\sigma_\beta(\pi^+\mathrm{P}\,|\,\theta^0\Lambda^{++})\ , \tag{2}$$

oltre alle solite relazioni che esprimono la più debole condizione di simmetria rispetto alla carica. La sezione d'urto per il processo che porta ai Λ^{++}, insieme a quella del processo simmetrico che porta ai Λ^-, sono le più grosse: in particolare se si vuole che esse si annullino tutte le altre sezioni d'urto devono parimenti annullarsi.

Caso γ: $T_\theta = 1$, $T_\Lambda = 1/2$, esistono θ^+, θ^0, θ^-, Λ^+ e Λ^0 (non c'è ancora posto per l'evento E di F.S.T.W. e non esistono particelle Λ^-). Intervengono due elementi di matrice, uno relativo allo stato con spin isotopico totale 1/2, l'altro relativo allo stato con spin isotopico totale 3/2, pertanto tre sole sezioni d'urto sono indipendenti a tutte le altre sono ad esse collegate tramite, ad esempio, le relazioni

$$\begin{aligned}\sigma_\gamma(\pi^+\mathrm{P}\,|\,\theta^+\Lambda^+) - 2\sigma_\gamma(\pi^0\mathrm{P}\,|\,\theta^0\Lambda^+) + \sigma_\gamma(\pi^-\mathrm{P}\,|\,\theta^-\Lambda^+) = \\ = \sigma_\gamma(\pi^-\mathrm{P}\,|\,\theta^0\Lambda^0) = \sigma_\gamma(\pi^0\mathrm{P}\,|\,\theta^+\Lambda^0)\ ,\end{aligned} \tag{3}$$

oltre alle solite relazioni che esprimono la più debole condizione di simmetria rispetto alla carica. A queste relazioni vanno aggiunte le disuguaglianze

$$3[\sigma_\gamma(\pi^-\mathrm{P}\,|\,\theta^-\Lambda^+) + \sigma_\gamma(\pi^-\mathrm{P}\,|\,\theta^0\Lambda^0)] \geqslant \sigma_\gamma(\pi^+\mathrm{P}\,|\,\theta^+\Lambda^+), \tag{3'}$$

$$\begin{aligned}[\sigma_\gamma(\pi^+\mathrm{P}\,|\,\theta^+\Lambda^+) + \sigma_\gamma(\pi^-\mathrm{P}\,|\,\theta^-\Lambda^+) - 2\sigma_\gamma(\pi^-\mathrm{P}\,|\,\theta^0\Lambda^0)]^2 \leqslant \\ \leqslant 4\sigma_\gamma(\pi^+\mathrm{P}\,|\,\theta^+\Lambda^+)\sigma_\gamma(\pi^-\mathrm{P}\,|\,\theta^-\Lambda^+)\ .\end{aligned} \tag{3''}$$

Le precedenti uguaglianze (3) e disuguaglianze (3′) e (3″) sono tutte e sole le condizioni richieste per la validità della indipendenza dalla carica, oltre alle solite relazioni che esprimono la più debole condizione di simmetria rispetto alla carica. Le disuguaglianze (3′) e (3″) derivano dalle uguaglianze (3) e dalla condizione che le sezioni d'urto siano positive definite al variare sul piano complesso degli elementi di matrice: pertanto le disuguaglianze (3′) e (3″) non derivano dalle uguaglianze (3) e dal fatto che le sezioni d'urto assumono *certi* valori positivi, e sono, a rigore, delle condizioni ulteriori che l'indipendenza dalla carica, se essa sussiste, impone ai risultati dell'esperienza. Dalle (3) si ricava, ad esempio, la disuguaglianza (debole)

$$\sigma_\gamma(\pi^-\mathrm{P}\,|\,\theta^0\Lambda^0) \leqslant \sigma_\gamma(\pi^+\mathrm{P}\,|\,\theta^+\Lambda^+) + \sigma_\gamma(\pi^-\mathrm{P}\,|\,\theta^-\Lambda^+)\,,$$

ove le sezioni d'urto che figurano possono tutte determinarsi con esperienze su idrogeno.

Caso δ: $T'_\theta = 1$, $T_\Lambda = 3/2$, esistono θ^+, θ^0, θ^-, Λ^{++}, Λ^+, Λ^0 e Λ^- (è il caso che meglio si adatterebbe alla attuale evidenza sperimentale, a parte la solita questione della esistenza di Λ^{++}). Intervengono due elementi di matrice, uno relativo allo stato con spin isotopico totale 1/2, l'altro relativo allo stato con spin isotopico totale 3/2, pertanto tre sole sezioni d'urto sono indipendenti e tutte le altre sono ad esse collegate tramite, ad esempio, le relazioni

$$(4)\quad \begin{cases} \sigma_\delta(\pi^+\mathrm{P}\,|\,\theta^0\Lambda^{++}) = -3\sigma_\delta(\pi^-\mathrm{P}\,|\,\theta^0\Lambda^0) + 3\sigma_\delta(\pi^-\mathrm{P}\,|\,\theta^-\Lambda^+) + \sigma_\delta(\pi^-\mathrm{P}\,|\,\theta^+\Lambda^-), \\ 2\sigma_\delta(\pi^0\mathrm{P}\,|\,\theta^-\Lambda^{++}) = 3\sigma_\delta(\pi^-\mathrm{P}\,|\,\theta^-\Lambda^+), \\ 3\sigma_\delta(\pi^+\mathrm{P}\,|\,\theta^+\Lambda^+) = -6\sigma_\delta(\pi^-\mathrm{P}\,|\,\theta^0\Lambda^0) + 6\sigma_\delta(\pi^-\mathrm{P}\,|\,\theta^-\Lambda^+) + 2\sigma_\delta(\pi^-\mathrm{P}\,|\,\theta^+\Lambda^-), \\ 3\sigma_\delta(\pi^0\mathrm{P}\,|\,\theta^0\Lambda^+) = \sigma_\delta(\pi^-\mathrm{P}\,|\,\theta^+\Lambda^-) \\ 2\sigma_\delta(\pi^0\mathrm{P}\,|\,\theta^+\Lambda^0) = 2\sigma_\delta(\pi^-\mathrm{P}\,|\,\theta^+\Lambda^-) + 3\sigma_\delta(\pi^-\mathrm{P}\,|\,\theta^-\Lambda^+) - 4\sigma_\delta(\pi^-\mathrm{P}\,|\,\theta^0\Lambda^0)\,, \end{cases}$$

oltre alle solite relazioni che esprimono la più debole condizione di simmetria rispetto alla carica. A queste relazioni vanno aggiunte le disuguaglianze

$$(4')\quad \begin{cases} 12\sigma_\delta(\pi^-\mathrm{P}\,|\,\theta^0\Lambda^0) + 2\sigma_\delta(\pi^-\mathrm{P}\,|\,\theta^+\Lambda^-) \geqslant 3\sigma_\delta(\pi^-\mathrm{P}\,|\,\theta^-\Lambda^+)\,, \\ [\sigma_\delta(\pi^-\mathrm{P}\,|\,\theta^+\Lambda^-) + 3\sigma_\delta(\pi^-\mathrm{P}\,|\,\theta^-\Lambda^+) - 6\sigma_\delta(\pi^-\mathrm{P}\,|\,\theta^0\Lambda^0)]^2 \leqslant \\ \qquad\qquad \leqslant 12\sigma_\delta(\pi^-\mathrm{P}\,|\,\theta^+\Lambda^-)\sigma_\delta(\pi^-\mathrm{P}\,|\,\theta^-\Lambda^+)\,, \end{cases}$$

che esprimono la condizione che le sezioni d'urto siano positive definite. Le precedenti uguaglianze (4) e disuguaglianze (4′) sono tutte e sole le condizioni richieste per la validità della indipendenza dalla carica, oltre alle solite rela-

zioni che esprimono la più debole condizione di simmetria rispetto alla carica. Si considerino, ad esempio, collisioni di pioni negativi su idrogeno (esperienza F.S.T.W.): dalle (4) si ricavano per le sezioni d'urto relative ai tre possibili processi le seguenti due disuguaglianze indipendenti

$$\sigma_\delta(\pi^-\mathrm{P}\,|\,\theta^0\Lambda^0) \leqslant (1/3)\sigma_\delta(\pi^-\mathrm{P}\,|\,\theta^+\Lambda^-) + \sigma_\delta(\pi^-\mathrm{P}\,|\,\theta^-\Lambda^+)\,,$$

$$\sigma_\delta(\pi^-\mathrm{P}\,|\,\theta^0\Lambda^0) \leqslant (1/2)\sigma_\delta(\pi^-\mathrm{P}\,|\,\theta^+\Lambda^-) + (3/4)\sigma_\delta(\pi^-\mathrm{P}\,|\,\theta^-\Lambda^+)\,.$$

Le precedenti disuguaglianze limitano ad ogni angolo la produzione di neutri (eventi A, B, D di S,F.T.W.) rispetto alla produzione di carichi (evento *E* di F.S.T.W.). Particelle Λ^{++} possono prodursi da collisioni di pioni positivi su idrogeno: dalle (4) segue che il processo di produzione di Λ^{++} deve essere 1,5 volte più frequente dell'altro processo possibile di produzione Λ^+. In generale, a partire dalle (4), è facile verificare che tutte le sezioni d'urto dei processi di produzione di Λ^0 (o di Λ^+) possono venire maggiorate con sezioni d'urto di processi di produzione di Λ^{++} (o, se si preferisce, con sezioni d'urto di processi simmetrici di produzione di Λ^-), ad esempio, nel seguente modo:

$$\sigma_\delta(\pi^-\mathrm{N}\,|\,\theta^-\Lambda^0) = (2/3)\sigma_\delta(\pi^+\mathrm{P}\,|\,\theta^0\Lambda^{++})\,,$$

$$\sigma_\delta(\pi^0\mathrm{N}\,|\,\theta^0\Lambda^0) = (1/3)\sigma_\delta(\pi^+\mathrm{N}\,|\,\theta^-\Lambda^{++})\,,$$

$$\sigma_\delta(\pi^+\mathrm{N}\,|\,\theta^+\Lambda^0) = (2/3)\sigma_\delta(\pi^0\mathrm{P}\,|\,\theta^-\Lambda^{++})\,,$$

$$\sigma_\delta(\pi^-\mathrm{P}\,|\,\theta^0\Lambda^0) \leqslant (2/3)\sigma_\delta(\pi^0\mathrm{P}\,|\,\theta^-\Lambda^{++}) + (1/3)\sigma_\delta(\pi^+\mathrm{N}\,|\,\theta^-\Lambda^{++})\,,$$

$$\sigma_\delta(\pi^0\mathrm{P}\,|\,\theta^+\Lambda^0) \leqslant (1/3)\sigma_\delta(\pi^+\mathrm{N}\,|\,\theta^-\Lambda^{++}) + (2/3)\sigma_\delta(\pi^+\mathrm{P}\,|\,\theta^0\Lambda^{++})\,,$$

In particolare, se si vuole che le sezioni d'urto dei processi di produzione di Λ^{++} si annullino, tutte le altre sezioni d'urto devono parimenti annullarsi. Questa conclusione resterebbe vera anche per altre assegnazioni di spin isotopico. Valori più elevati per lo spin isotopico della particella Λ contrasterebbero con il suo schema di decadimento (*se* si assume l'indipendenza dalla carica anche nei processi di decadimento); un valore 2 per lo spin isotopico del mesone pesante θ non contrasterebbe con il suo schema di decadimento, condurrebbe però a mesoni due volte carichi di cui finora non sussiste alcuna evidenza (per quanto la loro esistenza non può venire esclusa). Ulteriori informazioni sui processi, oltre queste fornite dalla indipendenza dalla carica richiederebbero ipotesi sugli elementi di matrice. In questo senso un caso molto particolare è il caso in cui la produzione avvenga attraverso stati isobarici intermedi dei nucleoni, a spin isotopico $T^* = 1/2$ oppure $T^* = 3/2$, come sembra essere il caso per altri fenomeni mesonici [7]. In questo caso tutti i

[7] Vedi, ad esempio, B. T. Feld: *Phys. Rev.*, **90**, 342 (1953).

rapporti tra le sezioni d'urto risulterebbero determinati in maniera univoca, e varrebbero le seguenti relazioni, in sostanza già contenute nelle precedenti (i precedenti casi α e β sono compatibili solo con $T^*= 1/2$ e $T^*= 3/2$ rispettivamente ed ovviamente valgono allora le stesse relazioni già ricavate)

$$\sigma^*_\gamma(\pi^+\mathrm{P}\,|\,\theta^+\Lambda^+) : \sigma^*_\gamma(\pi^0\mathrm{P}\,|\,\theta^0\Lambda^+) : \sigma^*_\gamma(\pi^-\mathrm{P}\,|\,\theta^-\Lambda^+) : \sigma^*_\gamma(\pi^-\mathrm{P}\,|\,\theta^0\Lambda^0) : \sigma^*_\gamma(\pi^0\mathrm{P}\,|\,\theta^+\Lambda^+) =$$

$$= 0 : 1 : 4 : 2 : 2 \qquad \text{per } T^* = 1/2;$$

$$= 9 : 4 : 1 : 2 : 2 \qquad \text{per } T^* = 3/2$$

$$\sigma^*_\delta(\pi^+\mathrm{P}\,|\,\theta^0\Lambda^{++}) : \sigma^*_\delta(\pi^0\mathrm{P}\,|\,\theta^-\Lambda^{++}) : \sigma^*_\delta(\pi^-\mathrm{P}\,|\,\theta^+\Lambda^-) : \sigma^*_\delta(\pi^+\mathrm{P}\,|\,\theta^+\Lambda^+) :$$

$$: \sigma^*_\delta(\pi^0\mathrm{P}\,|\,\theta^0\Lambda^+) : \sigma^*_\delta(\pi^-\mathrm{P}\,|\,\theta^-\Lambda^+) : \sigma^*_\delta(\pi^-\mathrm{P}\,|\,\theta^0\Lambda^0) : \sigma^*_\delta(\pi^0\mathrm{P}\,|\,\theta^+\Lambda^0) =$$

$$= 0 : 3 : 6 : 0 : 2 : 2 : 4 : 1\,, \qquad \text{per } T^* = 1/2;$$

$$= 27 : 12 : 6 : 18 : 2 : 8 : 1 : 16 \qquad \text{per } T^* = 3/2.$$

Tutte le relazioni ricavate finora restano ancora valide ove pure si scambino gli spin isotopici delle particelle finali, si assegni cioè lo spin isotopico intero alla particella Λ, lo spin isotopico semintero al mesone pesante (schema di Gell-Mann): basta contemporaneamente scambiare nelle formule ottenute le cariche dei prodotti finali (esempio $\sigma(\pi^+\mathrm{N}\,|\,\theta^+\Lambda^0)$ va in $\sigma(\pi^+\mathrm{N}\,|\,\theta^0\Lambda^+)$). Per vedere questo si ricordi la relazione $C_{jj'}(J, M; m, m') = (-1)^{j-j-j'} C_{j'j}(J, M; m', m)$: uno dei due elementi di matrice compare dunque con segno cambiato; ma le relazioni precedenti sono indipendenti dai valori assunti dagli elementi di matrice e, pertanto, rimangono invariate. Questa volta però Λ^0 e Λ^+ non sono più stati simmetrici rispetto alla carica. È facile però verificare che tutte le sezioni d'urto dei processi di produzione di Λ^0 possono venire maggiorate con sezioni d'urto di processi di produzione di Λ^- (o di Λ^+), ad esempio, nel seguente modo (nei casi α e β esistono solo Λ^0):

caso γ:

$$\sigma_\gamma(\pi^0\mathrm{P}\,|\,\theta^+\Lambda^0) \leqslant (1/2)[\sigma_\gamma(\pi^-\mathrm{N}\,|\,\theta^0\Lambda^-) + \sigma_\gamma(\pi^-\mathrm{P}\,|\,\theta^+\Lambda^-)],$$

$$\sigma_\gamma(\pi^+\mathrm{N}\,|\,\theta^+\Lambda^0) = \sigma_\gamma(\pi^0\mathrm{N}\,|\,\theta^+\Lambda^-)\,.$$

Caso δ:

$$\sigma_\delta(\pi^+\mathrm{P}\,|\,\theta^{++}\Lambda^0) \leqslant \sigma_\delta(\pi^+\mathrm{N}\,|\,\theta^{++}\Lambda^-) + 3\sigma_\delta(\pi^-\mathrm{P}\,|\,\theta^+\Lambda^-)\,,$$

$$\sigma_\delta(\pi^0\mathrm{P}\,|\,\theta^+\,\Lambda^0) = (1/3)\sigma_\delta(\pi^+\mathrm{N}\,|\,\theta^{++}\Lambda^-)\,,$$

$$\sigma_\delta(\pi^+\mathrm{N}\,|\,\theta^+\,\Lambda^0) \leqslant \sigma_\delta(\pi^-P\,|\,\theta^+\Lambda^-) + (1/3)\sigma_\delta(\pi^+\mathrm{N}\,|\,\theta^{++}\Lambda^-)\,.$$

In particolare, se si vuole che le sezioni d'urto dei processi di produzione di Λ^- (o di Λ^+) si annullino, tutte le altre sezioni d'urto devono parimenti annullarsi.

Del processo di produzione nucleone + nucleone → particella Λ + particella Λ non si ha ancora evidenza sperimentale. Tale processo, secondo PAIS, è il più semplice processo di produzione a partire da nucleoni, secondo GELL-MANN è ovviamente proibito, se si ammette la conservazione della componente z dello spin isotopico totale. [A questo punto possono farsi le seguenti osservazioni. Si consideri una teoria che assegni spin isotopici seminteri a tutti i fermioni e spin isotopici interi a tutti i bosoni; si considerino processi per cui l'indipendenza dalla carica è rispettata, *o, almeno, la componente z dello spin isotopico totale si conservi.* Condizione necessaria e sufficiente perchè la conservazione della carica (elettrica) risulti altresì verificata è che il numero (qui e nel seguito per numero intenderemo «numero delle particelle meno numero delle antiparticelle») dei fermioni iniziali sia uguale al numero dei fermioni finali. Due cose si possono quindi notare: primo, la conservazione della statistica risulta allora senz'altro soddisfatta e non comporta pertanto ulteriori regole di selezione; secondo, ove ci si limiti, come qui si immagina, a considerare processi in cui gli unici fermioni che intervengono o sono nucleoni o comunque appartengono alla famiglia dei nucleoni, la precedente condizione *garantisce, in quei processi, la conservazione delle particelle pesanti.* Si consideri ora una teoria (Gell-Mann) che distingue i fermioni in due classi: fermioni a spin isotopico semintero, che chiameremo vecchi fermioni, e fermioni a spin isotopico intero che chiameremo nuovi fermioni, e lo stesso per i bosoni: bosoni a spin isotopico intero, che chiameremo vecchi bosoni, e bosoni a spin isotopico semintero che chiameremo nuovi bosoni; si considerino processi per cui l'indipendenza dalla carica è rispettata, *o, almeno, la componente z dello spin isotopico totale si conservi.* Condizione necessaria e sufficiente perchè la conservazione della carica (elettrica) risulti altresì verificata è che il numero complessivo di vecchi fermioni e nuovi bosoni iniziali sia uguale al numero complessivo di vecchi fermioni e nuovi bosoni finali. A differenza di prima la conservazione della statistica e la conservazione delle particelle pesanti (nel senso già spiegato) non risultano contenute nella precedente condizione: vanno poste come condizioni ulteriori e comportano nuove regole di selezione. La conservazione della statistica richiede l'ulteriore condizione che sia pari il numero complessivo di nuove particelle, fermioni o bosoni, che intervengono in tutto il processo; *in particolare, richiede sempre la produzione a coppie.* La conservazione delle particelle pesanti richiede l'ulteriore condizione che resti costante la differenza tra il numero dei nuovi fermioni e il numero dei nuovi bosoni, e questa condizione contiene, come deve essere, la precedente. Viceversa se sono verificate, primo, la condizione che resti costante la differenza tra il numero dei nuovi fermioni e il numero dei nuovi bosoni, secondo, la conservazione delle particelle pesanti, ne segue che anche la conservazione della carica (elettrica) è verificata: *quindi, in particolare, il più generale processo di produzione partirà da* n ($\gtrless 0$) *vecchi fermioni* (*da pensarsi nucleoni*)

più un certo numero di vecchi bosoni (da pensarsi pioni) e giungerà ad $n - m$ ($\gtrless 0$) *vecchi fermioni, m nuovi fermioni, m nuovi bosoni, più un certo numero di vecchi bosoni.* Ad esempio, un processo nucleone + nucleone → particella Λ + particella Λ non rientra in questo schema, ed appunto perchè rispetta la conservazione delle particelle pesanti deve risultare proibito]. Il processo di produzione nucleone + nucleone → particella Λ + particella Λ verrà qui brevemente discusso nei due casi: caso ε, con $T_\Lambda = 1/2$; caso η, con $T_\Lambda = 3/2$.

Caso ε: intervengono due elementi di matrice, uno relativo allo stato di singoletto, l'altro relativo allo stato di tripletto; l'indipendenza dalla carica non dà luogo ad alcuna relazione tra le sezioni d'urto, oltre le solite che esprimono la più debole condizione di simmetria rispetto alla carica; la condizione che le sezioni d'urto siano positive definite risulta espressa dalle disuguaglianze

$$(5)\qquad \begin{cases} \sigma_\varepsilon(\mathrm{PP}\,|\,\Lambda^+\Lambda^+) \leqslant 2[\sigma_\varepsilon(\mathrm{PN}\,|\,\Lambda^+\Lambda^\circ) + \sigma_\varepsilon(\mathrm{PN}\,|\,\Lambda^\circ\Lambda^+)]\,, \\ [\sigma_\varepsilon(\mathrm{PP}\,|\,\Lambda^+\Lambda^+) - \sigma_\varepsilon(\mathrm{PN}\,|\,\Lambda^+\Lambda^\circ) - \sigma_\varepsilon(\mathrm{PN}\,|\,\Lambda^\circ\Lambda^+)]^2 \leqslant \\ \qquad\qquad \leqslant 4\sigma_\varepsilon(\mathrm{PN}\,|\,\Lambda^+\Lambda^\circ)\sigma_\varepsilon(\mathrm{PN}\,|\,\Lambda^\circ\Lambda^+)\,. \end{cases}$$

Caso η: intervengono due elementi di matrice, uno relativo agli stati di singoletto, l'altro relativo agli stati di tripletto, pertanto tre sole sezioni d'urto sono indipendenti e tutte le altre sono ad esse collegate tramite, ad esempio, le relazioni

$$(6)\qquad \begin{cases} 20\sigma_\eta(\mathrm{PP}\,|\,\Lambda^{++}\Lambda^\circ) = 15\sigma_\eta(\mathrm{PP}\,|\,\Lambda^+\Lambda^+) = 20\sigma_\eta(\mathrm{PP}\,|\,\Lambda^\circ\Lambda^{++})\,, \\ 3\sigma_\eta(\mathrm{PN}\,|\,\Lambda^+\Lambda^\circ) = -2\sigma_\eta(\mathrm{PP}\,|\,\Lambda^{++}\Lambda^\circ) + 2\sigma_\eta(\mathrm{PN}\,|\,\Lambda^{++}\Lambda^-) + \\ \qquad\qquad + \sigma_\eta(\mathrm{PN}\,|\,\Lambda^-\Lambda^{++})\,, \\ 3\sigma_\eta(\mathrm{PN}\,|\,\Lambda^\circ\Lambda^+) = -2\sigma_\eta(\mathrm{PP}\,|\,\Lambda^{++}\Lambda^\circ) + \sigma_\eta(\mathrm{PN}\,|\,\Lambda^{++}\Lambda^-) + 2\sigma_\eta(\mathrm{PN}\,|\,\Lambda^-\Lambda^{++}) \end{cases}$$

oltre alle solite relazioni che esprimono la più debole condizione di simmetria rispetto alla carica. A queste relazioni vanno aggiunte le disuguaglianze

$$(6')\qquad \begin{cases} 2[\sigma_\eta(\mathrm{PN}\,|\,\Lambda^{++}\Lambda^-) + \sigma_\eta(\mathrm{PN}\,|\,\Lambda^-\Lambda^{++})] \geqslant 3\sigma_\eta(\mathrm{PP}\,|\,\Lambda^{++}\Lambda^\circ)\,, \\ [3\sigma_\eta(\mathrm{PP}\,|\,\Lambda^{++}\Lambda^\circ) - \sigma_\eta(\mathrm{PN}\,|\,\Lambda^{++}\Lambda^-) - \sigma_\eta(\mathrm{PN}\,|\,\Lambda^-\Lambda^{++})]^2 \leqslant \\ \qquad\qquad \leqslant 4\sigma_\eta(\mathrm{PN}\,|\,\Lambda^{++}\Lambda^-)\sigma_\eta(\mathrm{PN}\,|\,\Lambda^-\Lambda^{++})\,, \end{cases}$$

che esprimono la condizione che le sezioni d'urto siano positive definite. Le precedenti uguaglianze (6) e disuguaglianze (6′) sono tutte e sole le condizioni

richieste per la validità della indipendenza dalla carica, oltre alle solite relazioni che esprimono la più debole condizione di simmetria rispetto alla carica. Si considerino, ad esempio, collisioni di protoni su neutroni, dalle (6) si ricavano

$$\sigma_\eta(\mathrm{PN}\,|\,\Lambda^{++}\Lambda^-) - \sigma_\eta(\mathrm{PN}\,|\,\Lambda^-\Lambda^{++}) = 3[\sigma_\eta(\mathrm{PN}\,|\,\Lambda^+\Lambda^\circ) - \sigma_\eta(\mathrm{PN}\,|\,\Lambda^\circ\Lambda^+)]\,,$$

$$\sigma_\eta(\mathrm{PN}\,|\,\Lambda^{++}\Lambda^-) + \sigma_\eta(\mathrm{PN}\,|\,\Lambda^-\Lambda^{++}) \geqslant \sigma_\eta(\mathrm{PN}\,|\,\Lambda^{++}\Lambda^\circ) + \sigma_\eta(\mathrm{PN}\,|\,\Lambda^\circ\Lambda^+)\,.$$

Le relazioni tra le sezioni d'urto totali (tanto nel caso ε quanto nel caso η) si ricavano semplicemente integrando le precedenti relazioni. Occorre però tenere presenti due punti: primo, processi che differiscono solo per l'ordine in cui figurano scritte le particelle o nello stato iniziale, o nello stato finale, o tanto nello stato iniziale quanto nello stato finale, e che in genere hanno diverse sezioni d'urto *differenziali*, ovviamente devono avere la stessa sezione d'urto *totale* [il termine d'interferenza singoletto-tripletto non contribuisce infatti alla sezione d'urto totale perchè antisimmetrico nelle coordinate spazio-spin delle particelle finali]; secondo, processi (differenziali) i cui stati finali differiscono solo per l'ordine in cui figurano scritte le particelle contribuiscono entrambi alla sezione d'urto totale solo e solo se le particelle *finali* sono distinguibili. Si ottiene allora (la barra indica sezione d'urto totale): caso ε: vale la disuguaglianza:

$$2\bar{\sigma}_\varepsilon(\mathrm{PN}\,|\,\Lambda^+\Lambda^\circ) \geqslant \bar{\sigma}_\varepsilon(\mathrm{PP}\,|\,\Lambda^+\Lambda^+)$$

più al solito le relazioni dovute alla simmetria rispetto alla carica; caso η: valgono le relazioni:

$$2\bar{\sigma}_\eta(\mathrm{PP}\,|\,\Lambda^{++}\Lambda^\circ) = 3\bar{\sigma}_\eta(\mathrm{PP}\,|\,\Lambda^+\Lambda^+)$$

$$3[\bar{\sigma}_\eta(\mathrm{PN}\,|\,\Lambda^{++}\Lambda^-) - \bar{\sigma}_\eta(\mathrm{PN}\,|\,\Lambda^+\Lambda^\circ)] = 2\bar{\sigma}_\eta(\mathrm{PP}\,|\,\Lambda^{++}\Lambda^\circ)$$

più al solito le relazioni dovute alla simmetria rispetto alla carica. Dalla seconda delle due disuguaglianze ora scritte si ricava in particolare

$$\bar{\sigma}_\eta(\mathrm{PN}\,|\,\Lambda^{++}\Lambda^-) \geqslant \bar{\sigma}_\eta(\mathrm{PN}\,|\,\Lambda^+\Lambda^\circ)\,.$$

Si vede pertanto che in collisioni protoni-protoni la produzione di Λ^{++} e Λ° è favorita per un fattore 1,5 rispetto alla produzione di Λ^+; in collisioni protoni-neutroni ancora la produzione di Λ^{++} e Λ^- è favorita rispetto alla produzione di Λ^+ e Λ°. In particolare, se si vuole che si annullino le sezioni d'urto dei processi di produzione di Λ^{++} tutte le altre sezioni d'urto devono parimenti annullarsi.

Si è già accennato alla possibilità che le interazioni (deboli) responsabili per il decadimento *non* rispettino l'indipendenza dalla carica. I processi di decadimento particella $\Lambda \to$ nucleone + pione, e mesone pesante $(\theta) \to$ pione +

+pione, risultano proibiti tanto nello schema di Pais (per la regola addizionale di parità), quanto nello schema di Gell-Mann. Questi due processi verranno qui brevemente discussi per quanto concerne l'indipendenza dalla carica intesa senza regole addizionali.

Decadimento particella $\Lambda \to$ nucleone + pione.

Caso ι: $T_\Lambda = 1/2$: vale la relazione

$$\sigma_\iota(\Lambda^0|\mathrm{P}\pi^-) = 2\sigma_\iota(\Lambda^0|\mathrm{N}\pi^0)$$

più le solite dovute alla simmetria rispetto alla carica.

Caso $\varkappa$: $T_\Lambda = 3/2$: valgono le relazioni (si noti la differenza col caso precedente)

$$2\sigma_\varkappa(\Lambda^-|\mathrm{N}\pi^-) = 3\sigma_\varkappa(\Lambda^0|\mathrm{N}\pi^0) = 6\sigma_\varkappa(\Lambda^0|\mathrm{P}\pi^-)$$

più le solite dovute alla simmetria rispetto alla carica.

Decadimento mesone pesante $(\theta) \to$ pione + pione.

Caso υ: $T_\theta = 0$: non vi sono direzioni privilegiate nello spazio dello spin isotopico, quindi

$$\sigma_\upsilon(\theta^0|\pi^0\pi^0) = \sigma_\upsilon(\theta^0|\pi^+\pi^-) = \sigma^\upsilon(\theta^0|\pi^-\pi^+)\,.$$

Per le sezioni d'urto totali si ha invece

$$2\bar{\sigma}_\upsilon(\theta^0|\pi^0\pi^0) = \bar{\sigma}_\upsilon(\theta^0|\pi^+\pi^-)\,.$$

Caso ω: $T_\theta = 1$: valgono le relazioni

$$\sigma_\omega(\theta^+|\pi^+\pi^0) = \sigma_\omega(\theta^+|\pi^0\pi^+) = \sigma_\omega(\theta^0|\pi^+\pi^-)\,,$$

$$\sigma_\omega(\theta^0|\pi^0\pi^0) = 0\,,$$

più le solite dovute alla simmetria rispetto alla carica. Per le sezioni d'urto totali

$$\bar{\sigma}_\omega(\theta^+|\pi^+\pi^0) = \bar{\sigma}_\omega(\theta^0|\pi^+\pi^-)\,; \qquad \bar{\sigma}_\omega(\theta^0|\pi^0\pi^0) = 0\,.$$

(Si noti la regola di selezione *debole* per decadimento in due pioni neutri).

Tutta la precedente discussione è stata svolta essenzialmente: primo, per fornire delle relazioni e delle disuguaglianze che potranno in seguito venire verificate, dato che processi semplici di produzione si sanno già possibili sperimentalmente; secondo, per studiare da vicino il meccanismo di tali processi

e vedere, ad esempio, *se con opportune ipotesi sugli ignoti elementi di matrice si poteva rendere conto della apparente scarsità di* Λ^{++}, *ed in genere di* Λ *carichi, contro* Λ *neutri: a questo riguardo la conclusione è che la indipendenza dalla carica esclude tale possibilità.* È superfluo aggiungere che la principale informazione sullo spin isotopico (eventuale) di queste particelle, potrebbe ovviamente ottenersi sapendo quali sono i loro stati di carica. Questo di per sè è già un problema sperimentalmente difficile, ma, comunque, il problema della validità della indipendenza dalla carica richiederebbe sempre l'ulteriore verifica di relazioni tra le sezioni d'urto del tipo di quelle riportate. Nel caso in cui nel processo elementare di produzione intervengano due elementi di matrice, di modo che i rapporti tra le sezioni d'urto non risultino tutti univocamente fissati dalla sola indipendenza dalla carica, può essere utile ricorrere a processi più complicati per verificare la indipendenza dalla carica ([1']). Per fare un esempio: nel processo (eventuale) nucleone + deutone → deutone + particella Λ + mesone pesante: nel caso γ ($T_\Lambda = 1/2$, $T_\theta = 1$) si ricava

$$\sigma_\gamma(\mathrm{Pd}\,|\,\Lambda^0 \mathrm{d}\theta^+) = 2\sigma_\gamma(\mathrm{Pd}\,|\,\Lambda^+ \mathrm{d}\theta^0)\,,$$

nel caso δ ($T_\Lambda = 3/2$, $T_\theta = 1$)

$$\sigma_\delta(\mathrm{Pd}\,|\,\Lambda^0 \mathrm{d}\theta^+) = (1/2)\sigma_\delta(\mathrm{Pd}\,|\,\Lambda^+ \mathrm{d}\theta^0) = (1/3)\sigma_\delta(\mathrm{Pd}\,|\,\Lambda^{++} \mathrm{d}\theta^-)\,,$$

[si noti la differenza rispetto al caso precedente]. La semplificazione proviene dal fatto che, come è noto, il deutone nel suo stato fondamentale ha spin isotopico zero.

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SUMMARY

We discuss the implications of the hypothesis of charge independence applied to simple processes of production of Λ particles, some of which have been recently observed. For completness we also discuss the decay processes of the particles involved, though the applicability of the charge independence hypothesis in this case may appear far less secure. Assuming half integral isotopic spin for the Λ particles, then, obviously, to any production process of Λ^0 particles there corresponds a production process of Λ^+ particles, having the same cross-section. Supposing moreover charge independence to

hold, it can be shown that the cross-sections for the production of Λ^0 (and Λ^+) particles can be severely limited in terms of the cross-section for the production of Λ^{++} (or Λ^-). Assuming integral isotopic spin for the Λ particles (GELL-MANN) it can still be shown that the cross-section for the production of Λ^0 particles can be severely limited in terms of the cross-section for the production of Λ^- (or Λ^+). On the other hand the Gell-Mann hypothesis, with a few reasonable assumptions, only allows the contemporary production of the same number (« number » we mean the difference between the number of particles and the number of antiparticles) of new Fermions and new Bosons, a prediction that should be soon capable of experimental verification. The particular relations and inequalities among the differential (and total) cross-sections, which result from the assumption of charge independence for the various assignments of isotopic spins, are given in the text and they may be of use to the experimentalists working in the field (of course, the other quite obvious relations which result from charge symmetry may be added). At any rate one would note that the reached conclusions very hardly could be reconciled with present experimental data, if one should decidedly interpret this data as giving evidence to a larger production of neutral Λ particles respect to charged ones (we particularly allude to Λ^{++} particles and to Λ^- particles).

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Total Ionization of α-Particles of Po in Mixtures of Gases.

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(ricevuto il 25 Febbraio 1954)

Summary. — Measurements have been made on the ionization of the α-particles of Po^{210} in mixtures of: A—H_2, A—N_2, A—CH_4, A—C_2H_5OH and A—C_6H_6. The plot of the ionization versus the concentration of the components of the mixture confirm the hypothesis of Haeberli and coworkers on the necessity of taking into account the ionization of δ-rays for the theoretical evaluation of the experimental results. They also confirm the hypothesis put forward by other researchers on the existence of an interaction between excited atoms of argon and the molecules of the foreign gas. In this process the molecule of the gas is ionized and an electron is collected in the chamber. The measurements were taken by means of a gridded ionization chamber and a fast recording chain that can amplify and analyse the individual pulses corresponding to the ionization of individual α-particles in the chamber.

1. – In the last few years numerous measurements have been effected on the ionization produced by α-particles in pure gas and mixtures of two gases. These measurements are interesting above all because they allow the investigation on both the mechanism of the ionization and the excitation of the molecules of the gas. In such experiments it has been established that the number of ions produced by an α-particle of a definite energy depends to a large extent on the purity of the gas itself. Recently W. P. Jesse and J. Sadauskis [1] have observed that the total ionization produced by α-particles of Po in helium and neon is increased by 43% and 39% respectively when small quantities of argon, about 0,1% are added to helium and neon. The values of the ionization obtained by the above researchers for He+0,1% of A and Ne+0,1% of A are in agreement with those hitherto accepted for the two pure gases. It is probable therefore that in all the preceding measurements of ioniz-

[1] W. P. Jesse and J. Sadauskis: *Phys. Rev.*, **88**, 417 (1952).

ation ([2-5]), the helium and neon used contained traces of argon. The difference in the results of the various experiments may depend to a great extent on the degree of purity of the gas and not on the particular technique used in the measurement.

J. SHARPE ([6]) has observed that the ionization in argon may be increased by about 1% by the admixture of 2 to 5% of CO_2 and O_2. It is interesting to note, and the reasons for this will be given later in this report, that the total ionization in oxygen and carbon dioxide is less than that in argon.

W. HAEBERLI, P. HUBER and E. BALDINGER ([7]) have effected analogous measurements in mixtures of A—O_2, A—He, N_2—O_2, A—N_2 and N_2—He; J. M. VALENTINE and S. G. CURRAN ([4]) in mixtures of A—CH_4, A—He, A—H_2 and A—N_2. These last researchers however ignore cases in which a component is contained in a quantity less than 10% in comparison with the other.

Our own measurements, referred to in this paper, are in respect of mixtures of A—H_2, A—N_2, A—CH_4, A—C_2H_5OH and A—C_6C_6 and take particularly into account measurements of ionizations in argon with small quantities of H_2, N_2, CH_4, C_2H_4OH and C_6H_6 (concentrations of a few units per mille to a few percentage units) ([8]).

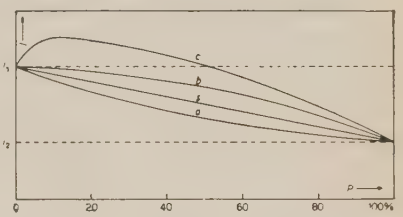

Fig. 1. – Different behaviours of the ionization in mixtures of two gases.

2. – We are illustrating in a graph the behaviour that can be expected for the mixtures and we shall discuss the physical processes that can produce such behaviour.

We now consider Fig. 1. On the X axis are shown the concentration of the foreign gas and on the Y axis the relative ionization. The segment s joins the two points corresponding to the ionization of the two pure components and represents the behaviour of the ionization according to the simple hypothesis that the two components behave indipendently from each other, as far as concerns the ionizing processes. The curves a, b, c, represent the three possible experimental behaviours that the various mixtures may have.

([2]) L. H. GRAY: *Proc. Cambr. Phys. Soc.*, **40**, 72 (1944).
([3]) G. STETTER: *Zeits. f. Phys.*, **120**, 639 (1943).
([4]) J. M. VALENTINE and S. G. CURRAN: *Phil. Mag.*, **43**, 964 (1952).
([5]) W. HAEBERLI, P. HUBER and E. BALDINGER: *Helv. Phys. Acta*, **25**, 467 (1952).
([6]) J. SHARPE: *Proc. Phys. Soc.*, A **65**, 859 (1952).
([7]) W. HAEBERLI, P. HUBER and E. BALDINGER: *Helv. Phys. Acta*, **23**, 481 (1950).
([8]) G. BERTOLINI, M. BETTONI and A. BISI: *Phys. Rev.*, **92**, 1586 (1953).

All the points of the curves *a* and *b* are comprised between the values of the ionization as measured for the two pure components. The deviation from the straight line can be explained by taking into account, in the calculation of the total ionization, the contribution of δ-rays generated from the α-particles themselves. The calculation of the total ionization has bee made by HAEBERLI and coworkers (9). The position of these curves whether above or below the segment *s* depends on the stopping powers of the δ-electrons and α-particles of the two gases and on the energy that the particles expend in producing the δ-rays.

The *c* type curves show an increase in the ionization with the increase of the concentration of a component. These curves are observed when processes of second order take place in the two gases and cause a transfer of energy between an excitated atom and an atom at ground state.

If we want to calculate the total ionization of one α-particle in a mixture of two gases it is necessary to take into account the way in which the energy is divided in which the energy of δ-rays is expended; and finally the transfer processes of energy from excited molecules to molecules at ground state.

When omitting the ionization produced by δ-rays, the calculation of HAEBERLI and coworkers lead to the following formula of the total ionization in a mixture of two gases:

$$\frac{I_M}{I_1} = \frac{W_1}{W_M} = \frac{s_1 p_1}{s_1 p_1 + s_2 p_2}\left(1 - \frac{W_1}{W_2}\right) + \frac{W_1}{W_2}, \tag{1}$$

where I_M = ionization of the mixture; I_1 and I_2 = ionization of the two gases; W_1, W_2 and W_M = average energy required to create a pair of ions in the two gases and in the mixture; p_1 and p_2 = partial pressure of the two gases; s_1 and s_2 = stopping powers of the two gases.

Formula (1) shows that W_1/W_2 is a linear function of the parameter

$$z = \frac{s_1 p_1}{s_1 p_1 + s_2 p_2}.$$

To take into account the ionization due to the secondary electrons, the following corrective term is added to the W_M value calculated according to (1):

$$\Delta(z) = W_1\left(\frac{1}{W_{s1}} - \frac{1}{W_{s2}}\right)(K_2 - K_1\gamma_{12})\frac{z(1-z)}{\gamma_{12} + z(1-\gamma_{12})}, \tag{2}$$

where

$$\gamma_{12} = \frac{s_1}{s_2}\frac{\sigma_2}{\sigma_1}.$$

(9) W. HAEBERLI, P. HUBER and E. BALDINGER: *Helv. Phys. Acta*, **26**, 145 (1953).

σ_1 and σ_2 indicate the average stopping powers for the electrons δ; W_1 and W_2 the average energy spent by the electrons δ to produce a pair of ions; K_1 and K_2 the fractions of energy E_1 and E_2 expended by an α-particle to produce electrons δ in the two gases.

3. – In the present work there was used a gridded ionization chamber ([10]) which permits the measurement of the electronic component of the pulse corresponding to the ionization of the α-particles. The chamber was connected to a gas purifier containing an alloy of calcium-magnesium which, as it was maintained at a temperature 300 to 400 °C, allowed the argon, or in some cases the mixtures under study, to circulate and be purified.

The mixture A—H_2 was purified by circulation on magnesium turnings in order to avoid the formation of calcium hydride. The alcohol and benzol used had a purity higher than 99,9%. Their vapours were introduced into the ionization chamber under vacuum and the argon, which had been under circulation, was added to them. Fig. 2 shows the schematic diagram of the filling and purifying apparatus.

The pressure of the mixture A—H_2 was 7 atm and that of the other mixture 3.5. With these values of the pressure, a complete collection of the electrons in all the mixtures studied was assured by applying a voltage between 2000 and 3500 V.

The source was prepared by evaporating a solution of Po^{210} on the negative electrode of the chamber.

The voltage pulse, taken off the positive collecting electrode connected to the ground by means of a 20 MΩ resistence, was amplified by a type 100 amplifier ([11]) and analyzed by a differential discriminator.

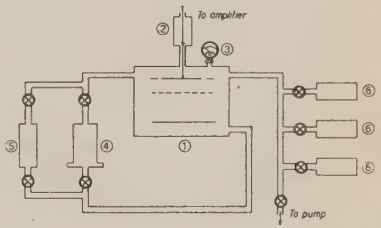

Fig. 2. – Schematic diagram of the filling and purifying apparatus. 1) ionization chamber; 2) pre-amplifier; 3) pressure gauge; 4) gas purifier; 5) mixture container; 6) gas tank.

For the purpose of studying the total ionization of α-particles in mixtures of gases, the characteristics of the amplifier are established by the different drift velocities of electrons in the different mixtures under study.

In a gridded ionization chamber the variation in the potential of the collector follows by some microseconds the absorption process of an α-particle in the collecting volume of the chamber. Such potential variation is character-

([10]) G. Bertolini, M. Bettoni and A. Bisi: *Nuovo Cimento*, **9**, 1004 (1052).

([11]) W. G. Elmore and M. Sands: *Electronics Experimental Tecniques* (New York, 1949), p. 165.

ised by its maximum value and by the time required to arrive at such value: this is called « the time rise » of the pulse and is given in a gridded chamber by the ratio between the distance from the grid to the collector and the drift velocity with which the center of gravity of the track moves towards the collector. Generally a linear amplifier amplifiyes differently the pulses with different rise time. The results of the ionization measured in different mixtures could therefore be altered in view of the different drift velocity of the electrons unless we suitably choose the two time constants which characterize the amplifier.

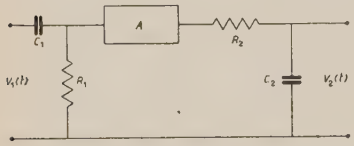

Fig. 3. – Equivalent circuit of a linear amplifier.

The behaviour of an amplifier can be determined with the aid of the equivalent circuit of Fig. 3; $R_1C_1 = \alpha$ and $R_2C_2 = \beta$ are the two time constants of high and low frequency respectively and A the amplifier stage.

The voltage pulse, collected on the electrode of a gridded ionization chamber (and generated by a point charge drifting across the chamber), has a linear rise and reaches the maximum value when the charge arrives at the collector ([12]).

In this case the height of the pulse at the output of the amplifier, can be calculated simply by using the two time constants α and β and the particular rise time of the pulse. The result of our calculation appears in Fig. 4.

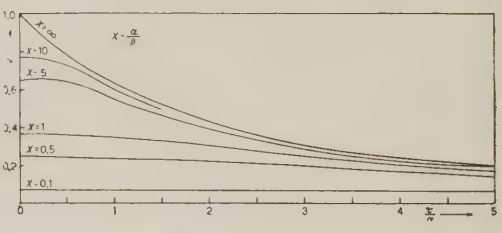

Fig. 4. – Height of an output pulse of a linear amplifier as a function of the rise time τ of the pulse. α and β are the amplifier's time constants. The height of the input signal is equal to 1.

As demonstrated in the graph it is possible to choose the time constants α and β in such a way as to render the height variation of the output pulses practically negligible even when τ varies noticeably. This choice is necessary firstly in order to get a most precise localization of the monochromatic line of Po^{210} on the energy axis, since the particles have a rise time that depends on the emission angle; and secondly in order to have an equal differentiation on the pulses indipendently of their rise time.

The amplifier type 100 has a rise time of 0.5 μs and a decay time of 100 μs and thus allows an equal amplification of the pulses of α-particles in the pure gases and in the mixtures under study. The values of the rise time of indi-

([12]) D. H. WILKINSON: *Ionization Chambers and Counters* (Cambdrige, 1950), p. 56.

vidual pulses were observed on the screen of a Du Mont oscillograph type 248B and are given in Table I.

TABLE I.

Gas	A	H_2	N_2	CH_4	A—H_2	A—N_2	A—CH_4	A—C_2H_5OH	A—C_6H_6
Rise Time of the Pulses (μs)	5	6	4	< 0,5	< 0,5	< 0,5	< 0,5	< 0,5	< 0,5

4. – The results of our measurements are indicated in Fig. 5, 6, 7, 8, 9, where the ratio between the ionization in the mixture and that in argon is plotted

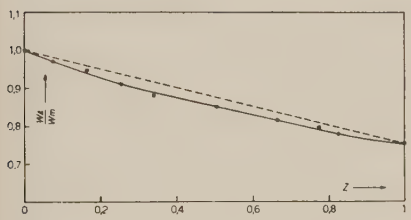

Fig. 5. – Mixture A—$N_2 \cdot W_A/W_M$ as function of z. Stopping powers $z_A = 0{,}95$, $s_{N_2} = 0{,}99$.

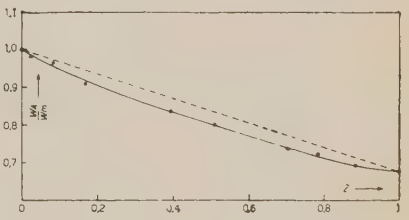

Fig. 6. – Mixture A—$H_2 \cdot W_A/W_M$ as function of z. Stopping powers $s_A = 0{,}95$, $s_{H_2} = 0.22$.

as a function of z. In the graph is also shown with a dotted line the same ratio according to (1).

In formula (1) we assume that the stopping powers are indipendent of the

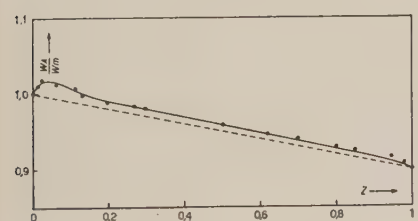

Fig. 7. – Mixture A—$CH_4 \cdot W_A/W_M$ as function of z. Stopping powers $s_A = 0.95$, $s_{CH_4} = 0.91$.

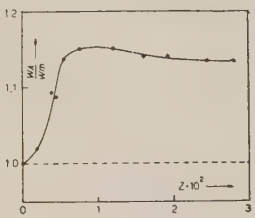

Fig. 8. – Mixture A—$C_2H_5OH \cdot W_A/W_M$ as function of z. Stopping powers $s_A = 0{,}95$ $s_{C_2H_5OH} = 2.02$.

energy of the α-particles; the values employed correspond to the total energy of the α-particles of Po ($E = 5.298$ MeV).

In the mixture A—CH_4, A—C_2H_5OH and A—C_6H_6 where the foreign gas

is in the order of 2%, 0,2% and 0,2% respectively, the ionization observed exceeds that of argon by 1,5%, 15% and 21%. It is to be recalled that in pure methane, as well as in alcohol and benzol, the ionization is inferior to that in argon ([13]).

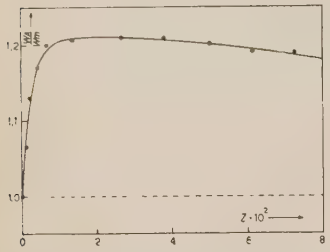

Fig. 9. – Mixture A—C_6H_6 · W_A/W_M as function of z. Stopping powers s_A=0.95, $s_{C_6H_6}$=3,33.

The results obtained for the mixtures A—H_2, A—N_2 and A—CH_4 coincide, within measurement errors, with those calculated by HAEBERLI and coworkers ([7]) and VALENTINE and CURRAN ([4]). In particular a comparison is made in Table II of the values obtained by various experimenters for the ionization in argon, nitrogen, hydrogen and methane.

TABLE II.

W_A/W_{N_2}	W_A/W_{H_2}	W_A/W_{CH_4}	Reference
0.727	0.775	—	GURNEY ([13])
0.765	0.787	—	STETTER ([3])
0.765	—	—	GILBERT *et al.* ([14])
0.723	—	—	HAEBERLI *et al.* ([7])
0.723	—	0.904	SHARPE ([6])
0.720	0.700	0.893	VALENTINE and CURRAN ([4])
0.757	0.675	0.900	Our measurements

5. – It can be seen from the graph that the value of N_M according to (1) agrees with the experimental results for the mixtures A—N_2 and A—H_2 only within a little percentage. A better approximation can be obtained by taking into account, as done by HAEBERLI and coworkers, the contribution of δ-rays; the correction $\Delta(z)$ calculated on the basis of equ. (2) is shown in Fig. 10-11 with the experimental points.

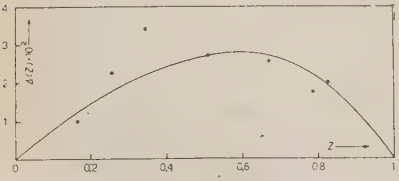

Fig. 10. – $\Delta(z)$ as calculated from formula (2) for A—N_2 mixture.

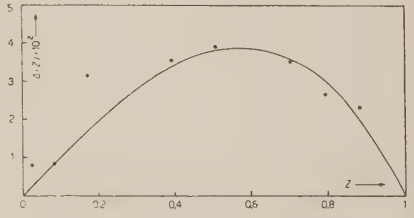

Fig. 11. – $\Delta(z)$ as calculated from formula (2) for A—H_2 mixture.

([13]) R. W. GURNEY: *Proc. Roy. Soc.*, **107**, 332 (1925).
([14]) A. GILBERT, P. ROGGEN and J. RUSSEL: *Helv. Phys. Acta*, **22**, 122 (1944).

For the above two mixtures, it is therefore possible to assume that no regaining of the energy expended by the α-particle to excite the molecule levels of the two components has taken place.

This can be explained by the fact that the ionization potential of the argon, nitrogen and hydrogen (15.68, 15.51 and 15.61 eV respectively) is greater than the excitation potential of any level of the argon, nitrogen and hydrogen.

For the mixtures A—CH_4, A—C_2H_5OH and A—C_6H_6 the experimental results suggest on the other hand that the following reaction has taken place:

$$A^* + S \rightarrow S^+ + A + e\,, \tag{3}$$

where S represents the components of the foreign gas (CH_4, C_2H_5OH, C_6H_6). The reaction (3) is possible because the ionization potentials of the methane, alcohol and benzol (14.5, 11.5 and 9.6 eV respectively) are inferior to the excitation potential of the argon levels (Table III) ([15]).

TABLE III.

Term	Excitation potential of Argon	Term	Excitation potential of Argon
$^1S_0$	0	$^1P_1$	11.825
$^3P_2$	11.545	$^{1,3}S, P, D$	12.7 ÷ 13.3
$^3P_1$	11.620	$^{1,3}P$	14.0 – 14.9
$^3P_0$	11.720	$^{1,3}P, D, F$	14.1 – 14.3

Fig. 12 shows the relative increase of the ionization versus the ionization potential of the component at small percentage. The levels which mostly contribute to such increase could be the following: $^3P_2$, $^3P_1$, $^3P_0$ and $^1P_1$; of which $^3P_2$ and $^3P_0$ are metastable.

The contribution to the total ionization given by the metastable levels excited by α-particles had been observed by JESSE and SADAUSKIS ([1]) in similar measurements of a mixture of helium and neon.

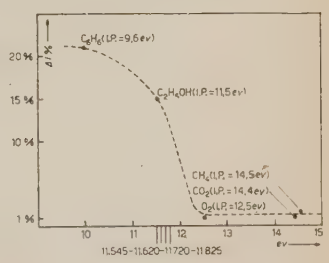

Fig. 12. – Percentage increase of the ionization as a function of the ionization potential of the foreign gas.

6. – In conclusion, our measurements confirm and extend the results of HAEBERLI and coworkers, for mixtures of a noble gas and a biatomic gas having approximatly the same ionization potential.

([15]) W. DE GROOT and F. M. PENNING: *Handbuch der Physik*, 23/1, p. 104; C. E. MOORE: *Atomic Energy Levels*, Circular NBS n. 4467 (1949).

Our measurements further show that in argon with a small percentage of a gas having an inferior ionization potential the ionization is considerably higher than in pure argon. The ionization increases as the ionization potential of the impurity decreases.

A plausible explanation for the increase in ionization observed is that an excited atom of argon transfers its energy to an atom of the impurity and ionizes it; all the levels, whether radiating or metastable, contribute, though in different degrees, to the total ionization.

Appendix

In order to define more fully the results shown in Fig. 4 we are giving the details of calculation of the height of the output pulse from a linear amplifier. Let $V_1(t)$ be the pulse from a gridded parallel-plate chamber

$$V_1(t)=\frac{1}{\tau}\{\, t1(t)-(t-\tau)1(t-\tau)\}\,. \tag{4}$$

In (4) $1(t)$ indicates the Dirichlet function and $1/\tau$ is introduced to normalize the $V_1(t)$.

The equivalent circuit of the amplifier is shown in Fig. 3. $R_1C_1=\alpha$ and $R_2C_2=\beta$ are the two time constants of high and low frequency respectively and A the amplifier stage. In this case the Laplace transformation of the output pulse can be written thus:

$$V_2(p)=\frac{1}{\tau}\frac{\alpha p}{1+\alpha p}\frac{1}{1+\beta p}\left\{\frac{1}{p^2}-\frac{e^{-p\tau}}{p^2}\right\}, \tag{5}$$

where

$$\frac{1}{\tau}\left\{\frac{1}{p^2}-\frac{e^{-p\tau}}{p^2}\right\},$$

indicates the transformation of the $V_1(t)$.

The inverse transformation of (5) does not present any difficulty and gives the following equation for the output pulse as a function of time:

$$V_2(t)=\frac{\alpha}{\tau}\frac{1}{\frac{1}{\alpha}-\frac{1}{\beta}}\left\{\frac{1}{\alpha}-\frac{1}{\beta}+\frac{1}{\beta}\exp[-t/\alpha]-\frac{1}{\alpha}\exp[-t/\beta]\right\}\qquad 0\leqslant t\leqslant\tau,$$

and

$$V_2(t)=\frac{\alpha}{\tau}\frac{1}{\frac{1}{\alpha}-\frac{1}{\beta}}\cdot$$

$$\cdot\left\{\frac{1}{\beta}(1-\exp[\tau/\alpha])\exp[-t/\alpha]-\frac{1}{\alpha}(1-\exp[\tau/\beta])\exp[-t/\beta]\right\}\qquad \tau\leqslant t\leqslant\infty.$$

The pulse reaches its maximum height in a time:

$$t = \frac{1}{\frac{1}{\alpha} - \frac{1}{\beta}} \log_e \frac{1 - \exp[\tau/\alpha]}{1 - \exp[\tau/\beta]} .$$

Correspondently the height of the pulse is

$$V_{2\,\max} = \frac{1}{y} (e^{xy} - 1)^{1/(1-x)} (e^{y} - 1)^{x/(x-1)},$$

where

$$x = \alpha/\beta \quad \text{and} \quad y = \tau/\alpha .$$

The dependence of $V_{2\,\max}$ on the rise time τ is shown for some values of the ratio α/β in Fig. 4.

We are greatly indebted to prof. G. Bolla for his encouragement, to Prof. B. Ferretti, Ing. E. Gatti and Dr. U. Facchini for helpful discussions.

RIASSUNTO

Vengono presentati i risultati di alcune misure sulla ionizzazione totale delle particelle α del Po^{210} in miscele di A—H_2, A—CH_4, A—C_2H_5OH e A—C_6H_6. L'andamento della ionizzazione in funzione delle percentuali dei due componenti della miscela conferma sia l'ipotesi di Haeberli e coll. sulla necessità di tener conto della ionizzazione dei raggi δ per la valutazione teorica dei risultati sperimentali, sia l'ipotesi affacciata da altri ricercatori sull'esistenza di una interazione tra atomi eccitati di argon, creati dall'α lungo il suo percorso, e molecole non eccitate dell'altro componente la miscela con conseguente ionizzazione di queste ultime. Le misure sono state eseguite mediante una camera di ionizzazione a griglia e catena di registrazione rapida, in modo da poter amplificare e analizzare i singoli impulsi di tensione corrispondenti alla ionizzazione delle singole α nella camera.

Construction of Potentials from Phase Shift and Binding Energies of Relativistic Equations.

E. Corinaldesi

Dublin Institute for Advanced Studies

(ricevuto il 25 Febbraio 1954)

Summary. — The problem of constructing the potential $V(r)$ from given S phase shift and binding energies of the Klein-Gordon equation is treated, and the analogue of the Gel'fand and Levitan integral equation is established.

Introduction.

A great advance in understanding the relation between a central potential and the phase shift and energy levels of the corresponding Schrödinger equation has recently been made by Jost and Kohn [1], who have applied to scattering theory the results of an extensive mathematical investigation of Gel'fand and Levitan [2]. Their method consists in determining the potential from the « spectral function », which embodies all the information derived from the knowledge of both phase shift and energies of the bound states for a given angular momentum. As such a spectral function contains a number of arbitrary parameters, as many as there are bound states, it is clear that a whole family of potentials can be constructed, which reproduce the given phase shift and binding energies.

In this paper our aim is to study the possibility of extending these methods to the case of relativistic equations. We shall take as an example the Klein-Gordon equation, for which the complications due to the spin do not arise, and which yet has two important features in common with the Dirac equation,

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[1] R. Jost and W. Kohn: *Kgl. Danske Vidensk. Selskab*, **27**, no. 9 (1953).

[2] I. M. Gel'fand and B. M. Levitan: *Dokl. Akad. Nauk SSSR*, **77**, 557 (1951); *Iz. Akad. Nauk SSSR*, **15**, 309 (1951).

which might prevent us from applying the procedure which has proved successful in the non-relativistic case. These are: *a*) the phase shift does not tend to zero when the energy tends to infinity, a consequence of the fact that the potential appears in the equation multiplied by the energy E; *b*) due to the occurrence of the two-valued quantity $E = \sqrt{k^2 + 1}$, the solutions of the Klein-Gordon equation (and, in general, of any relativistic equation) must be represented as functions of k on a Riemann surface. This has the effect that the potential can be constructed only if the phase shift is assigned for each branch of E. Moreover, the binding energies $-1 < E_l < +1$ are required. The phase shift for the negative branch of E, and the negative energy eigenvalues, must be interpreted physically as describing the scattering and the bound states of negative particles (3).

The assumption is made throughout this paper that the potentials with which we are dealing are not so strong that complex binding energies may occur. In fact, for very strong potentials the very use of the Klein-Gordon equation for the description of a single particle in a static field becomes meaningless (3).

We hope that the extension of this work to the case of the Dirac equation may constitute the subject of a subsequent paper.

1. – Preliminaries.

We consider the Klein-Gordon equation for S-states

$$\psi''(E, r) + k^2\psi(E, r) = (2EV - V^2)\psi(E, r)\,, \tag{1}$$

where $E = \sqrt{k^2 + 1}$. Following JOST and KOHN (1) (4), we define the solution $\varphi(E, r)$ with the boundary conditions

$$\varphi(E, 0) = 0\,, \qquad \varphi'(E, 0) = 1\,, \tag{2}$$

and the two linearly independent solutions $f(\pm k, r)$ with the asymptotic behaviour

$$\lim_{r\to\infty} \exp\,[\pm ikr]\, f(\pm k, r) = 1\,. \tag{3}$$

(3) H. KEMMER: *Proc. Roy. Soc.*, A **173**, 91 (1939); H. SNYDER and J. WEINBERG: *Phys. Rev.*, **57**, 307 (1940); L. I. SCHIFF, H. SNYDER and J. WEINBERG: *Phys. Rev.*, **57**, 315 (1940).

(4) R. JOST and W. KOHN: *Phys. Rev.*, **87**, 977 (1952)

The notation $f(k) \equiv f(k, 0)$ will be used extensively in the following. Note that, when k is real, $\varphi(E, r)$ can be expressed as [5]

$$\varphi\left(\sqrt{1+k^2}, r\right) = \frac{1}{2ik}\left[f(k)f(-k, r) - f(-k)f(k, r)\right]. \tag{4}$$

It is also convenient to introduce the function

$$\begin{cases} z = 2ik\,, \\ g(z, r) = g(2ik, r) = \exp\left[ikr\right]f(k, r)\,, \\ g(z) = g(z, 0)\,. \end{cases} \tag{5}$$

It is easily verified that the integral equation

$$g(z, r) = 1 + \int_r^\infty \frac{1}{z}\left[1 - \exp\left[-z(r'-r)\right]\right)\left(2EV(r') - \left(V(r')\right)^2\right)g(z, r')\,\mathrm{d}r', \tag{6}$$

is equivalent to eq. (1) for $f(k, r)$ with the boundary condition (3).

Eq. (6) can be solved by iteration for $\mathrm{Re}\,[z] \geqslant 0$ and for short range potentials with no pole at the origin. The resulting series can with advantage be used in order to show that

$$g(z, r) \sim \exp\left[i\varepsilon(\mu(r) - \mu(\infty))\right], \tag{7}$$

for $|z| \to \infty$ and $\mathrm{Re}\,[z] \geqslant 0$ [6]. In this formula

$$\begin{cases} \mu(r) = \int_0^r V(r')\,\mathrm{d}r'\,, \\ \nu(r) = \int_0^r \left(V(r')\right)^2\mathrm{d}r'\,, \end{cases} \tag{8}$$

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[5] In fact, it is evident that (4) satisfies eq. (1) and that $\varphi(\sqrt{1+k^2}, 0) = 0$. Moreover, $\varphi'(\sqrt{1+k^2}, 0) = 1$ is a consequence of the identity: $f(k, r)f'(-k, r) - f'(k, r)f(-k, r) = 2ik$, which holds also for the Schrödinger equation.

[6] Without entering into elaborate mathematical details, the asymptotic expression (7) can be justified by noticing that the $(n+1)$-th term of the series obtained by iterating eq. (6) is asymptotically equal to $\left(2E[\mu(\infty) - \mu(r)]\right)^n/n!\,z^n$. The symmetrization of the n-fold integral results from the fact that the contribution from the exponential term in the square bracket of eq. (16) vanishes in the limit $|z| \to \infty$ for $\mathrm{Re}[z] \geqslant 0$.

and $\varepsilon = E/k$. The occurrence of ε shows that we are dealing with a two-valued function of k.

From eqs. (5) and (7) we have, for $\mathrm{Im}[k] \leqslant 0$ and $|k| \to \infty$,

$$f(k, r) \sim \exp\left[-ikr + i\varepsilon(\mu(r) - \mu(\infty))\right]. \tag{9}$$

For $|k| \to \infty$ it is also true that

$$\varphi\left(\sqrt{1+k^2}, r\right) \sim \frac{1}{k}\sin\left(kr - \varepsilon\mu(r)\right). \tag{10}$$

This is a direct consequence of eqs. (4) and (9) if k is on the real axis, and can be shown to hold in the whole complex plane by using the integral equation for $\varphi(E, r)$ [7].

2. – Construcion of the function $f(k)$.

The S-matrix and the phase shift are now defined by

$$S(k) = \exp\left[2i\eta(k)\right] = \frac{f(k)}{f(-k)}, \qquad k \geqslant 0, \tag{11}$$

while the binding energies correspond to the zeros of the function $f(k)$ which lie in the half plane $\mathrm{Im}[k] \leqslant 0$ [8]. Such zeros will be denoted by k_l, so that $E_l = \sqrt{1 + k_l^2}$, and lie on the negative imaginary axis, if we assume that no complex binding energies occur.

All the functions so far defined, when extended to complex k, are two-valued. In the following we shall represent them on a Riemann surface obtained by cutting the k plane from $+i$ to $-i$ and by joining it cross-wise along the cut with another plane corresponding to the negative branch. The notation $\eta_+(k)$ and $\eta_-(k)$ (k real) will be used to designate the phase shift corresponding to the positive and negative branches, respectively. The physical meaning of these quantities has been stressed in the introduction.

The relation

$$\eta_\pm(k) = \mathrm{Im}\left[\log g_\pm(z)\right], \qquad (k \text{ real}), \tag{12}$$

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(7) The proof can be constructed by analogy with that for the non-relativistic case. Cf., e.g., P. Jauho: *Ann. Acad. Sci. Fennicae*, Ser. A, **80** (1951).

(8) It should be noted that what is considered here are the zeros of $f(k)$, which always mean bound states, and not of $S(k)$, which has also redundant zeros due to an infinity of $f(-k)$.

in which we use the self-explanatory notation $g_{\pm}$, is a direct consequence of eqs. (5) and (11). Moreover,

$$\lim_{k\to\infty} \eta_{\pm}(k) = \mp\, \mu(\infty) \tag{13}$$

results from eq. (7).

The phase shift can be defined also for negative values of k, by noticing that, being $f(-k) = f^*(k)$ for real k, eq. (12) yields $\eta_{\pm}(k)+\eta_{\pm}(-k) = 2n\pi$. Here we may put $n = 0$, thus assuming the phase shift to be an odd function of k on the real axis for each branch.

In analogy with the non-relativistic case, we must now show that the functions $g_{\pm}(z)$ - $f_{\pm}(k)$ can be constructed from the phase shift and the binding energies. From eq. (12) we write

$$\eta_+(k) + \eta_-(k) = \mathrm{Im}\left[\log\left(g_+(z)g_-(z)\right)\right], \tag{14}$$

where both sides are one-valued functions of k. The expression on the right hand side vanishes for $|z| \to \infty$ and $\mathrm{Re}[z] \geqslant 0$. Thus, if no bound states exist, we can invert eq. (14) by Poisson's integral

$$\log\left(g_+(z)g_-(z)\right) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\eta_+(k') + \eta_-(k')}{k - k'}\,\mathrm{d}k' . \tag{15}$$

A similar argument enables us to write

$$\frac{1}{\varepsilon(k)}\log\left(\frac{g_+(z)}{g_-(z)}\right) + 2i\mu(\infty) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{d}k'}{k - k'}\left[2\mu(\infty) + \frac{\eta_+(k') - \eta_-(k')}{\varepsilon(k')}\right], \tag{16}$$

also in the absence of bound states. The two above equations determine $g_+(z)$ and $g_-(z)$. The procedure to be followed in the case when there are bound states is identical with that given by Jost and Kohn [4] for the Schrödinger equation, and will not be repeated here.

3. – The spectral function.

In the non-relativistic theory, the « spectral function » $\varrho_s(E)$ is defined as the weight factor occurring in the expansion $F(r) = \int a(E)\varphi_s(E, r)\,\mathrm{d}\varrho(E)$ of a function $F(r)$ in terms of the eigenfunctions of the Schrödinger equation. As the latter form a complete orthonormal set, the expansion coefficients $a(E)$ are uniquely determined.

This is not the case for the Klein-Gordon equation, whose eigenfunctions do not constitute a complete orthonormal set in the usual sense, as they satisfy

only the quasi-orthogonality relation

$$\int_0^\infty (E + E' - 2V(r))\varphi(E, r)\varphi(E', r)\,\mathrm{d}r = 0 \qquad (E \neq E')\,. \tag{17}$$

Therefore the expansion of $F(r)$ in terms of $\varphi(E, r)$ is not sufficient to determine the coefficients $a(E)$.

This point has been investigated by various authors, and the results are summarized in a paper by DYSON (9).

From the eigenfunctions $\varphi(E, r)$ he constructs the two-component functions

$$\Phi(E, r) = \begin{pmatrix} \varphi(E, r) \\ i\,(E - V(r))\varphi(E, r) \end{pmatrix}, \tag{18}$$

which form a complete orthogonal set in a metric defined by the scalar product

$$(\Psi_1, \Psi_2) = \int_0^\infty \Psi_1^\dagger \gamma \Psi_2\,\mathrm{d}r\,. \tag{19}$$

Here γ is the Hermitean matrix $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and the dagger denotes the adjoint. Then any suitably restricted function $F \equiv \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$ has the unique expansion

$$F(r) = \int \mathrm{d}\varrho(E)(\Phi(E), F)\Phi(E, r)\,, \tag{20}$$

which completely defines the spectral function $\varrho(E)$.

For E between $+1$ and -1, $\varrho(E)$ is explicitly given by the equation

$$\frac{\mathrm{d}\varrho}{\mathrm{d}E} = \sum_l C_l \delta(E - E_l)\,, \tag{21}$$

where $C_l = \left[2\int_0^\infty (E_l - V)\varphi_l^2 \mathrm{d}r\right]^{-1}$, and E_l and φ_l denote the energy and the eigenfunctions of a bound state.

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(9) F. J. DYSON: *Phys. Rev.*, **73**, 617 (1948), where reference is made to the work by SNYDER and WEINBERG (3). Two component equations equivalent to the Klein-Gordon equation were given by SAKATA and TAKETANI: *Scient. Papers of the Inst. of Phys. and Chem. Research*, Tokyo, **38**, 1 (1940); W. HEITLER: *Proc. Roy. Irish Acad.*, **49**, no. 1 (1943).

For $|E| \geqslant 1$, adapting to the Klein-Gordon equation the classical method of normalization in the continuous spectrum ([10]), it is easy to find that

$$\frac{\mathrm{d}\varrho}{\mathrm{d}E} = \frac{k}{\pi |f(k)|^2} . \tag{22}$$

The spectral function refers, of course, to both branches of E. It may happen that no bound states exist for either one or both the branches.

4. – The integral equation.

Let $V_1(r)$ denote an arbitrary auxiliary potential with corresponding solutions $\varphi_1(E, r)$, phase shift $\eta_{\pm}^{(1)}(k)$ and binding energies $E_{1l}^{\pm}$ for positive and negative particles. The spectral function $\varrho_1(E)$ is completely determined, $f_1(k)$ and C_{1l} being known quantities.

Let $V(r)$ be an unknown potential which produces the (known) phase shift $\eta_{\pm}(k)$ and a number $m^{\pm}$ of bound states with the (known) energies $E_l^{\pm}$ for positive and negative particles respectively. The function $f(k)$ can then be constructed as shown in sect. **2**, and the spectral function $\varrho(E)$ determined, but for the $m^+ + m^-$ parameters C_l.

We can now establish the equation

$$\varphi(E, r) = \varphi_1(E, r) \cos(\mu(r) - \mu_1(r)) + \int_0^r [K_1(r, t) + EK_0(r, t)] \varphi_1(E, t) \, \mathrm{d}t , \tag{23}$$

where

$$\begin{cases} K_0(r, t) = \int \mathrm{d}[\varrho_1(E') - \varrho(E')] \, \varphi(E', r) \varphi_1(E', t) , \\ K_1(r, t) = \int \mathrm{d}[\varrho_1(E') - \varrho(E')] \varphi(E', r) \varphi_1(E', t) (E' - 2V_1(t)) . \end{cases} \tag{24}$$

The proof is similar to that for the non-relativistic case ([1]), and will be given in the appendix. The integral relation (23), however, differs from its non-relativistic analogue in two respects. Firstly, the factor $\cos(\mu(r) - \mu_1(r))$ occurs on the right hand side. This is a consequence of the non-vanishing value of the phase shift at infinite energy. Secondly, the combination $K_1(r, t) + EK_0(r, t)$ occurs instead of a sole function $K(r, t)$. This is due to the quasi-orthogonality of the Klein-Gordon wave functions, expressed by eq. (17).

([10]) A. Sommerfeld: *Wave-Mechanics* (London, 1930), p. 293.

In order to derive the analogue of the Gel'fand and Levitan integral equation, we multiply eq. (23) by $\varphi_1(E, s)$ and integrate over E with the weight $\mathrm{d}[\varrho(E) - \varrho_1(E)]$. We obtain

$$0 = K_0(r, s) + P_0(r, s)\cos(\mu(r) - \mu_1(r)) + \int_0^r [(K_1(r, t) + 2V_1(t)K_0(r, t))P_0(t, s) + K_0(r, t)P_1(t, s)]\,\mathrm{d}t\,, \tag{25}$$

where

$$\begin{cases} P_0(r, t) = \int \varphi_1(E', r)\varphi_1(E', t)\,\mathrm{d}[\varrho(E') - \varrho_1(E')]\,, \\ P_1(r, t) = \int \varphi_1(E', r)\varphi_1(E', t)(E' - 2V_1(t))\,\mathrm{d}[\varrho(E') - \varrho_1(E')]\,. \end{cases} \tag{26}$$

On the other hand, multiplying by $\varphi_1(E, s)(E - 2V_1(s))$ and also integrating over E with the weight $\mathrm{d}[\varrho(E) - \varrho_1(E)]$ we get

$$0 = K_1(r, s) + P_1(r, s)\cos(\mu(r) - \mu_1(r)) + \int_0^r [(K_1(r, t) + 2V_1(t)K_0(r, t))P_1(t, s) + K_0(r, t)P_2(t, s)]\,\mathrm{d}t\,, \tag{27}$$

where

$$P_2(r, t) = \int \varphi_1(E', r)\varphi_1(E', t)(E' - 2V_1(r))(E' - 2V_1(t))\,\mathrm{d}[\varrho(E') - \varrho_1(E')]\,. \tag{28}$$

Eqs. (25) and (27) can be used in order to calculate the functions

$$\begin{cases} \mathfrak{K}_0(r, t) = K_0(r, t)/\cos((\mu r) - \mu_1(r))\,, \\ \mathfrak{K}_1(r, t) = K_1(r, t)/\cos(\mu(r) - \mu_1(r))\,. \end{cases} \tag{29}$$

The functions $P_0(r, t)$, $P_1(r, t)$ and $P_2(r, t)$ are entirely determined by the scattering data and binding energies relative to $V(r)$ and by the auxiliary potential $V_1(r)$.

By a calculation similar to that yielding eq. (23) (and which will also be given in the appendix), it is not difficult to show that

$$\mathfrak{K}_0(r, r) = -2i\,\mathrm{tg}\,(\mu(r) - \mu_1(r))\,, \tag{30}$$

This equation can be reduced to a very simple form by differentiation with respect to r [11]

$$V(r) - V_1(r) = \frac{\mathrm{d}}{\mathrm{d}r} \operatorname{tg}^{-1}\left(\frac{i\Re_0(r, r)}{2}\right), \tag{31}$$

Thus, once eqs. (25) and (27) are solved, the potential $V(r)$, which reproduces the phase shift $\eta_{\pm}(k)$ and the binding energies $E_i^{\pm}$, is readily found from eq. (31). Such a potential depends on the $m^+ + m^-$ undetermined parameters C_i, which occur in the spectral function $\varrho(E)$ and therefore also in $P_0(r, t)$, $P_1(r, t)$ and $P_2(r, t)$. As in the non-relativistic case so also for the Klein-Gordon equation there exists a family of potentials with the same phase shift and binding energies for both positive and negative particles. If only the phase shifts are given, the position of the bound states is completely arbitrary.

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It is a pleasure to express my thanks to Prof. W. Heitler for his interest in this work and for reading and criticizing the manuscript.

Appendix

1. – Derivation of eq. (23).

The integral on the right hand side is equal to the difference $J - I$, where

$$I = \int \mathrm{d}\varrho(E')\, \varphi(E', r) \int_0^r \varphi_1(E', t)\varphi_1(E, t)(E + E' - 2V_1(t))\, \mathrm{d}t\,, \tag{A.1}$$

$$J = \int \mathrm{d}\varrho_1(E')(E', r) \int_0^r \varphi_1(E', t)\varphi_1(E, t)(E + E' - 2V_1(t))\, \mathrm{d}t\,. \tag{A.2}$$

[11] For comparison, note that, in the non-relativistic case, the relation between the potentials $V(r)$ and $V_1(r)$ and the function $K(r, t)$ is $\frac{1}{2}\left[V(r) - V_1(r)\right] = \frac{\mathrm{d}}{\mathrm{d}r} K(r, r)$.

We first evaluate the integral I. According to eqs. (21) and (22), we can write

$$I = \sum_l C_l \varphi(E_l, r) \int_0^r \varphi_1(E_l, t)\varphi_1(E, t)(E + E_l - 2V_1(t))\mathrm{d}t + \tag{A.3}$$

$$+ \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{k'^2\,\mathrm{d}k'}{E' f(k') f(-k')}\varphi(E', r)\int_0^r \varphi_1(E', t)\varphi_1(E, t)(E + E' - 2V_1(t))\,\mathrm{d}t\,.$$

The sum and the integration are extended to both branches of E_l and E' respectively.

Use must now be made of

$$\varphi(E_l, r) = \frac{f(k_l, r)}{f'(k_l, 0)}\,, \tag{A.4}$$

and

$$\int_0^\infty (E_l - V)\varphi_l^2\,\mathrm{d}r = -\frac{E_l \dot{f}(k_l, 0)}{2k_l f'(k_l, 0)}\,, \tag{A.5}$$

where f' and $\dot{f}$ denote differentiation with respect to r and k respectively. The first of these relations results from the extension of eq. (4) to complex values of k, together with $f(k_l) = 0$ and the differential identity given in footnote [5]. The second can be established if we integrate $f(k, r)^2 \cdot (\partial/\partial k) \cdot (f''(k, r)/f(k, r))$ over r from 0 to ∞ and use eq. (1), together with $f(k_l) = 0$ and eq. (A.4). Employing eqs. (A.4) and (A.5) it is easy to show that the integral along the real axis k' can be reduced to a sum of residues which cancel the first term and an integral over a semicircle of infinite radius in the upper half plane, which can be evaluated by using the asymptotic expressions (9) and (10). Thus we obtain

$$I = \frac{1}{2}\varphi_1(E, r)\cos\big(\mu(r) - \mu_1(r)\big)\,. \tag{A.6}$$

The evaluation of J runs along similar lines. Use is first made of the relation

$$\int_0^r (E + E' - 2V_1(t))\,\varphi_1(E', t)\varphi_1(E, t)\,\mathrm{d}t = \tag{A.7}$$

$$= \frac{\varphi_1(E, r)\varphi_1'(E', r) - \varphi_1'(E, r)\varphi_1(E', r)}{E - E'}\,,$$

which can easily be derived from the Klein-Gordon equation. The integral with respect to E' is resolved into a contribution from the bound states and an integral over k' from $-\infty$ to $+\infty$. The latter is then transformed into an integral over a large semicircle in the upper half plane, a sum of residues at the points $k = k_l$ which cancels the contribution from the bound states,

and a further residue at $E' = E$. Evaluating the integral over the large semicircle, again by the use of the asymptotic expressions (9) and (10), we obtain

$$J = \varphi(E, r) - \frac{1}{2}\varphi_1(E, r)\cos\left(\mu(r) - \mu_1(r)\right). \tag{A.8}$$

This establishes eq. (23).

2. – Derivation of eq. (30).

In contrast with the procedure followed in reference ([1]), we prefer to evaluate $K_0(r, r)$ by contour integration. This can be written

$$K_0(r, r) = \tag{A.9}$$
$$= \frac{1}{2\pi i}\int \frac{k'\,\mathrm{d}k'}{E'}\left[\frac{f_1(-k', r)}{f_1(\quad k')}\varphi(\sqrt{1+k'^2}, r) - \frac{f(-k', r)}{f(\quad k')}\varphi_1(\sqrt{1+k'^2}, r)\right],$$

where the integral is extended over a large semicircle as in the previous sect., and a sum of residues has cancelled the contribution from the bound states. Using the asymptotic expressions (9) and (10) and adding the contributions from the two branches of E' we obtain

$$K_0(r, r) = -2i\sin\left(\mu(r) - \mu_1(r)\right). \tag{A.10}$$

RIASSUNTO (*)

Si tratta il problema della costruzione del potenziale $V(r)$ a partire da un dato spostamento di fase S e dalle energie di legame dell'equazione di Klein-Gordon, e si determina un'equazione analoga all'equazione integrale di Gel'fand e Levitan.

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A Study of τ-Meson Decay.

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Summary. — The distribution of the π-mesons in the decay of the τ-meson is studied, and approximate expressions for the matrix elements are found. Thus one can justify Dalitz' results, in which the relativistic corrections are also introduced. The comparison with experiment is done taking into account the charges of the π-mesons; the experimental data, however, are still too poor to give any information about parity and spin of the τ-meson.

The angular distribution and energy spectrum of π-mesons emitted in the decay of the τ-meson have been investigated recently by Dalitz [1], with the aim of obtaining information about the spin and parity of the τ-meson. Dalitz' work is based on an analysis of the dependence of the decay amplitude on the momenta of the emitted particles. The essential hypothesis is that this dependence is the simplest of all those compatible with a given spin and parity of the τ-meson which allows the angular and energy distribution of the π-mesons to be uniquely determined in 5 cases which will be indicated later. Another hypothesis of Dalitz is that the maximum value of the energy of the π-meson (about 50 MeV) is low enough to allow a non-relativistic treatment.

Since up to recently it was not experimentally possible to determine the charge of the π-mesons emitted in τ-decay, it was then found expedient to try to obtain as much information as possible independently of the charge as Dalitz has done. The recent development of stripped emulsion technique makes it appear probable that in the near future the majority of observed

(1) R. H. Dalitz: *Phil. Mag.*, **44**, 1068 (1953).

τ-decay will allow a determination of the charges of the π-mesons; it will therefore be convenient to take account of these as well, which, as will be seen, makes discrimination between the various hypotheses considerably easier.

In this paper, the distribution in the τ-decay will therefore be re-studied with the following aims: 1) to confirm Dalitz' essential hypothesis, which seems the weakest point in his work; 2) to introduce the relativistic corrections which, as will be seen, are not always negligible; 3) to present the results in the most convenient form for comparison with experimental data.

The treatment which follows is divided into four sections: in the first we give an analysis of the system of the three π-mesons from a kinematic point of view; in the second the τ-decay is examined in the light of the conservation theorems of quantum mechanics; in the third we study the dependence of the decay matrix element on the parameters of the final state; finally in the fourth section the results are set out in the most convenient form for comparison with experimental work. The third section also contains the proof of a result recently published [2] on the mean life of the τ-meson.

1. – Kinematic analysis of the three meson system.

In the following let m be the mass of the π-meson, E the total kinetic energy in the CM system [3] (i.e. the Q of the decay) $\boldsymbol{r}_i$, $\boldsymbol{p}_i$ $(i = 1, 2, 3)$ the radius vectors and momenta of the three mesons (we take the mesons 1 and 2 to have charges equal and opposite to that of 3); and E_i the total energies, always in the CM system.

In place of the variables $\boldsymbol{r}_i$ we introduce

$$(1) \qquad \begin{cases} \boldsymbol{R} = \dfrac{1}{\sqrt{3}}(\boldsymbol{r}_1 + \boldsymbol{r}_2 + \boldsymbol{r}_3) \\ \boldsymbol{r} = \dfrac{1}{\sqrt{2}}(\boldsymbol{r}_2 - \boldsymbol{r}_1) \\ \boldsymbol{r}' = \sqrt{\dfrac{2}{3}}\left[\boldsymbol{r}_3 - \dfrac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_2)\right]. \end{cases}$$

It is immediately verified that the transformation is linear and orthogonal, so that the momenta conjugate to the new variables will be obtained by the

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[2] E. FABRI and B. F. TOUSCHEK: *Nuovo Cimento*, **11**, 96 (1954).

[3] In this paper the reference system with respect to which the total momentum is zero will throughout be called « the centre of mass (CM) system ».

same transformation law:

$$(2)\qquad \begin{cases} \boldsymbol{P} = \dfrac{1}{\sqrt{3}}\,(\boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3) \\ \boldsymbol{p} = \dfrac{1}{\sqrt{2}}\,(\boldsymbol{p}_2 - \boldsymbol{p}_1) \\ \boldsymbol{p}' = \sqrt{\dfrac{2}{3}}\left[\boldsymbol{p}_3 - \dfrac{1}{2}\,(\boldsymbol{p}_1 + \boldsymbol{p}_2)\right]. \end{cases}$$

The CM system is defined by $\boldsymbol{P} = 0$ (4), and from (1) and (2) it is easily verified that the angular momentum in this system is

$$(3)\qquad \boldsymbol{K} = \boldsymbol{k} + \boldsymbol{k}', \qquad \text{with} \qquad \boldsymbol{k} = \boldsymbol{r}\times\boldsymbol{p}, \quad \boldsymbol{k}' = \boldsymbol{r}'\times\boldsymbol{p}'.$$

As regards $\boldsymbol{p}$ and $\boldsymbol{p}'$, they are convenient variables for expressing the asymptotic distribution of the three particles; to do this it will be in fact as well to separate the moduli p and p' (which are not independent because of the conservation of energy) from the unit vectors $\boldsymbol{u}$ and $\boldsymbol{u}'$ which give the angular distribution.

2. – Consequences of the conservation theorems in τ-decay.

In τ-decay, momentum, energy and angular momentum are conserved; to these first integrals we can add the total parity, and the isotopic spin if its

(4) In quantum mechanics, equations (1), as also (2), form a complete system of commuting variables and it does not seem correct simply to put $\boldsymbol{P} = 0$ to pass into the CM system. In order to do this, only states of the dynamical system corresponding to the eigenvalue zero of the observable $\boldsymbol{P}$ must be considered; this shows the convenience of assuming $\boldsymbol{P}$ as one of the basic observables.

It is appropriate at this point to clarify the meaning of the expression « angular momentum in the CM system » from the point of view of quantum mechanics. Because of (1) and (2) the observable « total angular momentum » has the form:

$$(*)\qquad \boldsymbol{K} + \boldsymbol{R}\times\boldsymbol{P}$$

and it can easily be shown that $|\boldsymbol{K}+\boldsymbol{R}\times\boldsymbol{P}|^2$ does not commute with $\boldsymbol{P}$. According to the interpretation given above of the CM system, we should limit ourselves to considering states corresponding to the eigenvalue zero of $\boldsymbol{P}$; but it is evident that the matrix element of (*) between two such states is identical with that of $\boldsymbol{K}$. Furthermore the commutator of $|\boldsymbol{K}+\boldsymbol{R}\times\boldsymbol{P}|^2$ with $\boldsymbol{P}$ has eigenvalue zero for the states considered, so that it is possible to assign a definite value to the modulus of the total angular momentum. Thus it can be concluded that it is compatible with the principles of quantum mechanics to put $\boldsymbol{P}=0$ when referring to the CM system.

existence is admitted also for the τ-meson. The isotopic spin is of no immediate interest in this work, since its conservation only furnishes relations between the probabilities of the two concurrent processes $\tau^{\pm} \to 2\pi^{\pm} + \pi^{\mp}$ and $\tau^{-} \to 2\pi^{0} + \pi^{\pm}$ which are of the same type as those found by DALITZ [5], and gives no direct information on the decay of the first type.

A consequence of the conservation of momentum is that the CM system of the three π-mesons coincides with the restsystem of the τ-meson, which is obviously the most natural frame of reference. As regards the conservatiou of energy, we will not deal with it at the moment. Thus there remain angular momentum and total parity, which are known as soon as a definite spin and parity have been assigned to the τ-meson. The subject of this section will be, in fact, the connection between the spin and parity of the τ-meson and the angular distribution of the final π-mesons.

Having fixed a unit vector $\boldsymbol{u}_0$, we will in the following denote the eigenvalues of $|\boldsymbol{K}|^2$ by $J(J+1)$ [6] and those of $\boldsymbol{u}_0 \cdot \boldsymbol{K}$ by j. An analogous meaning with respect to $\boldsymbol{k}$ and $\boldsymbol{k}'$ will be given to $l(l+1)$, m and to $l'(l'+1)$, m' respectively. The final state, which can be expressed as a function of the variables p, $\boldsymbol{u}$, $\boldsymbol{u}'$, will be an eigenstate of $|\boldsymbol{K}|^2$ and $\boldsymbol{u}_0 \cdot \boldsymbol{K}$ relative to certain values of J and j corresponding to the spin of the τ-meson and its orientation in space; if then the parity of the τ-meson is denoted by $P(\pm 1)$ the final state must have parity $-P$ because of the presence of three π-mesons with intrinsic parity -1. If account is taken of the identity of the mesons 1 and 2 it is then seen that the final state is even with respect to $\boldsymbol{u}$, so that to a parity P of the τ-meson there corresponds a parity $-P$ of the final state with respect to $\boldsymbol{u}'$.

We now see that it is possible to expand the final state in simultaneous eigenstates of $|\boldsymbol{K}|^2$, $\boldsymbol{u}_0 \cdot \boldsymbol{K}$, $|\boldsymbol{k}|^2$, $|\boldsymbol{k}'|^2$ in the form

$$\psi(p\,|\,\boldsymbol{u}, \boldsymbol{u}') = \sum_{ll'} c_{ll'}(p) Z_{ll'}^{(J,j)}(\boldsymbol{u}, \boldsymbol{u}')\,, \tag{5}$$

where $Z_{ll'}^{(J,j)}$ has the form:

$$Z_{ll'}^{(J,j)}(\boldsymbol{u}, \boldsymbol{u}') = \sum_{m,m'} s_{J,m,m'}^{(l,l')} \gamma_{lm}(\boldsymbol{u}) \gamma_{l'm'}(\boldsymbol{u}') \delta_{j,m+m'}\,, \tag{6}$$

($s_{J,m,m'}^{(l,l')}$ are the Clebsch-Gordan coefficients in the Wigner notation [7]).

The sum in (5) is limited to values of l, l', which satisfy the relation $|l - l'| \leqslant J \leqslant l + l'$; there is yet another limitation due to the parity. From the preceding it is found, in fact, that l must be even, while l' is even or odd

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(5) R. H. DALITZ: *Proc. Phys. Soc.*, **66**, 710 (1953).

(6) We will always put $c = 1$, $\hbar = 1$.

(7) E. WIGNER: *Gruppentheorie* (Braunschweig, 1931).

according as $P=-1$ or $P=+1$. From this there follow for l, l' the possibilities indicated in Table I.

On the left of the dotted line are the pairs (l, l') corresponding to the minimum value of $l+l'$ compatible with each choice of J and P. It can be seen that no pair (l, l') exists for $J=0$, $P=+1$, which expresses the fact that in this case decay into three π-mesons is not possible. Again we note that there is a unique pair (l, l') with minimum $l+l'$ in 5 cases: [0—], [1+], [1—], [2+], [3—] [8]; we will make use of this later [9].

TABLE I.

J	P	(l, l')	
0	+1	—	—
	−1	(0,0)	(2,2); (4,4); .....
1	+1	(0,1)	(2,1); (4,3); .....
	−1	(2,2)	(4,4); (6,6); .....
2	+1	(2,1)	(2,3); (4,3); .....
	−1	(0,2); (2,0)	(2,4); (4,2); .....
3	+1	(0,3); (2,1)	(2,3); (4,1); .....
	−1	(2,2)	(2,4); (4,2); .....
4	+1	(2,3); (4,1)	(2,5); (4,3); .....
	−1	(0,4); (2,2); (4,0)	(2,4); (4,2); .....
.....		.....	.....

When it is known how to calculate the coefficients $c_{ll'}$, equation (5) gives the space and energy distribution of the π-mesons; experimentally, (as long as the τ-mesons can not be polarized) we are only interested in the distribution in the plane of the three vectors $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, $\boldsymbol{p}_3$. Therefore it will be convenient to replace the variables $\boldsymbol{u}$, $\boldsymbol{u}'$ by the angle ϑ defined by $\cos\vartheta = \boldsymbol{u}\cdot\boldsymbol{u}'$, which, together with p, is sufficient to define uniquely the momentum triangle.

[8] The symbol [1 —] represents the hypothesis $J=0$, $P=1$; and similarly for the other cases.

[9] We note in passing that these 5 are just those cases in which DALITZ gives a definite distribution; the reason for this lies in the strong connection between the values of l and l' and the dependence of the matrix element on p and p'.

It is now necessary to perform the integration

$$F(p,\vartheta)=\int |\psi(p\,|\boldsymbol{u},\boldsymbol{u}')|^2\,\delta(\boldsymbol{u}\cdot\boldsymbol{u}'-\cos\vartheta)\,\mathrm{d}\boldsymbol{u}\,\mathrm{d}\boldsymbol{u}'\,, \tag{7}$$

which gives the desired distribution. The calculation can be simplified by observing that the same result is obtained by integrating solely with respect to $\boldsymbol{u}'$ and then summing with respect to j; one can therefore put $\boldsymbol{u}=\boldsymbol{u}_0$ and obtain:

$$F(p,\vartheta)=\sum_j{}'\sum_{ll'} c_{ll'}(p)\cdot s^{(ll')}_{J,0,j}\left\{(2l+1)(2l'+1)\frac{(l'-|j|)!}{(l'+|j|)!}\right\}^{\frac{1}{2}}\cdot P_{l'}^{|j|}(\cos\vartheta)\Big|^2. \tag{8}$$

The procedure followed is correct in so far as the statistical weight of the final state obviously does not depend on $\boldsymbol{u}$ and $\boldsymbol{u}'$ except through ϑ. In a non-relativistic theory it would be found immediately that this weight depends solely on p, while this is not the case in the exact theory. From this it follows that in general it is not correct to calculate the distribution in p from the formula

$$\sum_{ll'}|c_{ll'}(p)|^2\,. \tag{9}$$

This result would be obtained in the non-relativistic case, integrating the absolute square of the right hand side of equation (5) with respect to $\boldsymbol{u}$ and $\boldsymbol{u}'$ (the functions $Z^{(J,j)}_{ll'}$) are orthogonal). In fact equation (9) becomes more complicated by the occurrence of interference terms. One can also account for this fact from another point of view: in non-relativistic approximation the total kinetic energy is $(p^2+p'^2)/2m$ and so commutes with $|\boldsymbol{k}|^2$ and $|\boldsymbol{k}'|^2$; this justifies formula (9) because it is possible to have a final state with l and l' defined. In the relativistic expression for the energy, however, there appears a term $\boldsymbol{p}\cdot\boldsymbol{p}'$ which is the cause on the one hand of the dependence of the statistical factor on ϑ, and on the other makes the energy not commute with $|\boldsymbol{k}|^2$ and $|\boldsymbol{k}'|^2$, which explains the appearance of the interference terms.

3. – Determination of the coefficients $c_{ll'}(p)$.

Formula (8) reduces the determination of the distribution under discussion to the determination of the coefficients $c_{ll'}(p)$; we will now give an approximate evaluation of these coefficients, which will allow us a considerable simplification in the use of (8); we will also give the order of magnitude of the errors introduced by this approximation and conditions for its validity.

The idea to be developed has already been explained in [2]; as was in-

dicated in the introduction, we can consider the coefficients $c_{ll'}$ as the elements of the S-matrix between the initial and final states defined as follows: the initial state is a τ-meson with spin J, parity P and component of spin j in the direction of $\boldsymbol{u}_0$; the meson is at rest in the reference system considered. The final state is defined by the values l and l' of the two relative angular momenta of the three π-mesons, as well as by the quantum numbers J, j, P which are the same as for the initial state, and finally by the parameters p, p'. It is clear that $c_{ll'}$ does not depend on j, and so we can write:

$$c_{ll'}(p) = a \int \mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{r}' \cdot f^{(j)}(\boldsymbol{r}, \boldsymbol{r}') \cdot j_l(pr) \cdot j_{l'}(p'r') \cdot Z_{ll'}^{(J,j)}(\boldsymbol{v}, \boldsymbol{v}') \,, \tag{10}$$

where in $f^{(j)}(\boldsymbol{r}, \boldsymbol{r}')$ appear the nucleus of the integral expression of the S-matrix and the wave function of the initial state, integrated over the coordinates of the τ-meson. Naturally f can depend only on the relative coordinates of the π-mesons, that is on $\boldsymbol{r}$ and $\boldsymbol{r}'$. The constant a depends on J and P but not on l and l', j_l is the spherical Bessel function of order l; $\boldsymbol{v}$, $\boldsymbol{v}'$ are the unit-vectors of $\boldsymbol{r}$, $\boldsymbol{r}'$. Integrating the right hand side with respect to $\boldsymbol{v}$, $\boldsymbol{v}'$, (10) becomes:

$$c_{ll'}(p) = a \int \mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{r}' \cdot f^0_{ll'}(r, r') \cdot j_l(pr) \cdot j_{l'}(p'r') \,, \tag{11}$$

where the meaning of $f^0_{ll'}$ is obvious.

(11) is an exact expression, in which we must now introduce two quite independent approximations. The first, which we will discuss later, consists in substituting for j_l the first term of the series expansion of r, that is $[1/(2l+1)!!](pr)^l$ (and the same for $j_{l'}$). The second consists in a hypothesis on the form of $f^0_{ll'}(r, r')$. If Coulomb interaction is ignored, it is logical to suppose that $f^{(j)}$ depends symmetrically on $\boldsymbol{r}_1 - \boldsymbol{r}_2$, $\boldsymbol{r}_2 - \boldsymbol{r}_3$, $\boldsymbol{r}_3 - \boldsymbol{r}_1$; therefore we will admit that $f^0_{ll'}$ depends only on $|\boldsymbol{r}_1 - \boldsymbol{r}_2|^2 + |\boldsymbol{r}_2 - \boldsymbol{r}_3|^2 + |\boldsymbol{r}_3 - \boldsymbol{r}_1|^2 = 3(r^2 + r'^2) = 3s^2$. With $r = s\cos\alpha$, $r' = s\sin\alpha$, (11) becomes

$$\begin{aligned} c_{ll'}(p) &= a \frac{1}{(2l+1)!!} \frac{1}{(2l'+1)!!} p^l p'^{l'} \cdot (4\pi)^2 \int_0^\infty s^{l+l'+5} g(s)\mathrm{d}s \cdot \\ &\cdot \int_0^{2\pi} \sin^{l'+2}\alpha \cdot \cos^{l+2}\alpha \cdot \mathrm{d}\alpha = \\ &= (4\pi)^2 a \cdot \frac{1}{(2l+1)!!} \frac{1}{(2l'+1)!!} p^l p'^{l'} \cdot \frac{(l+1)!!\,(l'+1)!!}{(l+l'+4)!!} \varepsilon_{l+l'} \int_0^\infty s^{l+l'+5} g(s)\,\mathrm{d}s \,, \end{aligned} \tag{12}$$

where

$$\varepsilon_n = \begin{cases} 2\pi & \text{for even } n \,, \\ 4 & \text{for odd } n \,. \end{cases}$$

If the function $g(s)$, which replaces $f^0_{ll'}(r, r')$, does not depend on l, l', we can consider it to be normalized so that $\int_0^\infty s^5 g(s)\,ds = 1$. If $g(s)$ is then negligible when $s > 1/M$ (M mass of the τ-meson), which means that there is no sensible interaction at a distance greater than the Compton wavelength of the τ-meson, then one will certainly have:

$$\int_0^\infty s^{\lambda+5} g(s)\,ds = \frac{\alpha_\lambda}{M^\lambda}, \tag{13}$$

where α_λ is a coefficient of modulus less than 1 which decreases as λ increases.

Concluding, the expression:

$$c_{ll'}(p) = A\,\frac{\alpha_{l+l'}}{(l+l'+4)!!}\cdot\left(\frac{m}{M}\right)^{l+l'}\cdot\frac{(l+1)!!}{(2l+1)!!}\cdot\frac{(l'+1)!!}{(2l'+1)!!}\cdot\left(\frac{p}{m}\right)^{l}\cdot\left(\frac{p'}{m}\right)^{l'}, \tag{14}$$

is obtained for $c_{ll'}$ where A depends only upon J and P.

Bearing in mind that both p and p' are always less than m (see section **4**) it can be said that for a given initial state only the $c_{ll'}$ with the minimum $l+l'$ are important; all the other $c_{ll'}$ are in fact smaller by a factor of at least 100.

The considerations developed allow a notable simplification of equation (8) which reduces to a justification of Dalitz' idea of considering only the simplest cases for the dependence on the $\boldsymbol{p}_i$; in particular we are now in a position to write explicitly the distributions in p, p', ϑ for the 5 cases [0—], [1+], [1—], [2+], [3—], (see Table II) (10).

TABLE II.

[0 —]	1
[1 +]	p'^2
[1 —]	$p^4 p'^4 \sin^2\vartheta\cos^2\vartheta$
[2 +]	$p^4 p'^2 \sin^2\vartheta$
[3 —]	$p^4 p'^4 \sin^2\vartheta\,(5 + 3\cos^2\vartheta)$

Another important consequence of (14) is the possibility, shown in (2), of explaining the mean life of the τ-meson in the hypothesis [1—]. In fact the

(10) Without going into details we observe that the distributions in Table II can be obtained using Dalitz' methods, but rather more simply, by considering the two vectors $\boldsymbol{p}$, $\boldsymbol{p}'$ instead of the three $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, $\boldsymbol{p}_3$. In fact the matrix element must be even in $\boldsymbol{p}$, and even or odd in $\boldsymbol{p}'$ according as $P = -1$ or $+1$. Forming with the vectors $\boldsymbol{p}$ and $\boldsymbol{p}'$ the irreducible tensors of degree J, with the condition that $\boldsymbol{p}$ appears an even number of times, and $\boldsymbol{p}'$ an even or odd number of times according to the value of P, one immediately recovers the results of Table II.

modulus square of c_{00} integrated with respect to p gives the reciprocal of the mean life on the hypothesis [0 —], while the same operation performed on c_{22} gives the corresponding value for the case [1 —]. It is seen, performing the calculations, that the ratio between the two mean lives is $\sim 10^{12}$; supposing then that for the case [0 —] the mean life is not very different from $1/M$, that is, of the order of $10^{-23} - 10^{-21}$ s, we would have for the hypothesis [1 —] a value of about $10^{-11} - 10^{-9}$ s. Agreement with experimental data is satisfactory, at least as regards order of magnitude.

We will conclude this section with a discussion of the approximations made and of their possible consequences.

Let us examine in the first place the evaluation of the integral (13), which may seem at first sight quite arbitrary. In fact it can easily be demonstrated that if $g(s)$ is not identically zero for $s > 1/M$, the coefficient α_λ, instead of decreasing with λ, will certainly diverge. However, it should be borne in mind that the factor s^λ in (13), which is the cause of the divergence, comes from the substitution for $j_l(pr)$ and $j_{l'}(p'r')$ of their lower order terms in r and r'; this will be correct only for $pr \ll l$ and $p'r' \ll l'$, so that strictly speaking a power series with a similar behaviour to that of j_λ and coefficients depending on p and p' should appear in equation (13) instead of s^λ. In these conditions the justification of equation (13) appears to be sufficiently well-founded, and the order of the error, which is about 2%, can also be given for it.

The other point to be discussed is the form of the dependence of $f^0_{ll'}$ on r, r'. In fact this form cannot be completely justified, but it must be observed that a different form for the $f^0_{ll'}$ would give results qualitatively similar; in fact, what we need is only a way of evaluating the various $c_{ll'}$ and of showing that some are negligible in comparison with others. It could be observed that the major danger is that $f^0_{ll'}$ depends very strongly on l, l'; in fact a precise physical hypothesis is needed here: we suppose that $f^{(j)}(\boldsymbol{r}, \boldsymbol{r}')$ is appreciably different from zero in a region of the space $\boldsymbol{r}$, $\boldsymbol{r}'$, whose size is always of the order of $1/M$ whatever the angle between $\boldsymbol{r}$ and $\boldsymbol{r}'$ may be. From this point of view the procedure followed has the sole justification of allowing a simple enough deduction of the required results; another form of $f^0_{ll'}$, provided it satisfies the hypothesis just given, would not give anything new.

4. – The Energy Distribution.

Table II summarizes the results of the work done so far; however, it is not of immediate use, since the variables p, p', ϑ, in which the expected distributions are expressed, have no simple connection with the quantities which can actually be measured. We now want to reformulate our results in such a way that they may be compared directly with experimental data.

We have, from conservation of energy:

$$E_1 + E_2 + E_3 = M = 3m + E\,, \tag{15}$$

where $E = 76$ MeV. In the following we will write $\varepsilon = E/M = 0{,}154$, from which it follows that $m = M(1-\varepsilon)/3$. Two parameters are sufficient, as has been seen, to represent the τ-decay from a kinematic point of view, and so it is possible to set up a one to one correspondence between the possible final states and the points of a plane region. This can be done in several ways; we shall adopt the well known representation (already used by DALITZ) which consists in interpreting the distance of the representative point from the sides of an equilateral triangle as kinetic energies of the three mesons. Assuming a system of polar coordinates (ϱ, φ) with pole in the centre of the triangle and polar axis passing through a vertex, the following equations are obtained:

$$\left\{\begin{aligned} E_1 &= \frac{M}{3}\left[1 + \varepsilon\varrho\,\cos\left(\varphi - \frac{2\pi}{3}\right)\right], \\ E_2 &= \frac{M}{3}\left[1 + \varepsilon\varrho\,\cos\left(\varphi + \frac{2\pi}{3}\right)\right], \\ E_3 &= \frac{M}{3}\left[1 + \varepsilon\varrho\,\cos\varphi\right]. \end{aligned}\right. \tag{16}$$

Equations (16) show that the unit of length is chosen so that the distance of P from the side $\overline{12}$ is $(E_3 - m)/(E/3)$, etc.

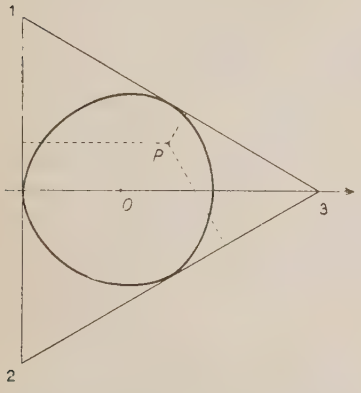

Fig. 1.

We see at once that from the conservation of energy alone the whole of the triangle is permitted, while the conservation of momentum imposes a further restriction, that is that a triangle can be formed with sides p_1, p_2, p_3. Taking account of the relations $E_i^2 = m^2 + p_i^2$ and equations (16), the above mentioned condition becomes:

$$\varrho^2 \leqslant \frac{1}{1 + \alpha(1 + \varrho\,\cos 3\varphi)}\,, \tag{17}$$

where $\alpha = \frac{1}{2}\,\varepsilon/(1 - \frac{1}{2}\varepsilon)^2 = 0.09$.

It is clear from the meaning of ε that the non-relativistic limit is obtained for $\varepsilon \to 0$, and thus for $\alpha \to 0$; in this case (17) becomes simply $\varrho^2 \leqslant 1$, which is the circle inscribed in the triangle. If relativistic corrections are taken into account, the boundary of the permit-

ted region becomes somewhat restricted, though still remaining tangential to the sides of the triangle at their midponts; it is easily found that the minimum distance from the centre is obtained when $q = 0, \pm 2\pi/3$, and is ~ 0.92.

As regards the statistical weight, we have [11]:

$$E_1 E_2 E_3 \, dE_1 \, dE_2 , \tag{18}$$

which is transformed by (16) (apart from constants) into:

$$\left[1 + \frac{3}{4}\varepsilon^2\varrho^2\left(1 + \frac{1}{3}\varepsilon\varrho\cos 3\varphi\right)\right] \varrho \, d\varrho \, d\varphi. \tag{19}$$

The factor in square brackets in (19) reduces to 1 for $\varepsilon = 0$, but even for the actual value of ε never exceeds 1.02.

To obtain the distributions in ϱ and φ it is enough sufficient at this point to express p, p', ϑ in terms of the new variables; this can easily be done observing that from equations (2) for $\boldsymbol{P} = 0$ are obtained

$$\begin{cases} p^2 = p_1^2 + p_2^2 - \dfrac{1}{2} p_3^2 \\ p'^2 = \dfrac{3}{2} p_3^2 \\ pp' \cos\vartheta = \boldsymbol{p}\cdot\boldsymbol{p}' = \dfrac{\sqrt{3}}{2}(p_1^2 - p_2^2) . \end{cases} \tag{20}$$

Expressing p_i in terms of E_i and substituting equations (16) again, we obtain:

$$\begin{cases} p^2 = \dfrac{1}{3} M^2\varepsilon\left[(1 - \varrho\cos\varphi) - \dfrac{1}{2}\varepsilon(1 - \varrho^2\sin^2\varphi)\right] \\ p'^2 = \dfrac{1}{3} M^2\varepsilon(1 + \varrho\cos\varphi)\left[1 - \dfrac{1}{2}\varepsilon(1 - \varrho\cos\varphi)\right] \\ pp'\cos\vartheta = \dfrac{1}{3} M^2\varepsilon\cdot\varrho\sin\varphi\left(1 - \dfrac{1}{2}\varepsilon\cos\varphi\right). \end{cases} \tag{21}$$

It is now possible to calculate the distribution function $\Phi(\varrho, \varphi)$ from the data given in Table II expressed in terms of (21) and multiplied by the factor (19); however we will not give the results of these calculations here, since at present they could hardly be compared with experiment. In effect the most reasonable way of carrying out the comparison with experimental data is to divide the permitted region into a certain number of partial regions, and then compare the calculated probability of the regions thus obtained

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[11] E. Amaldi, G. Baroni, C. Castagnoli, G. Cortini amd A. Manfredini: *Nuovo Cimento*, **10**, 937 (1953).

with the observed frequency. The problem to be solved now is the following: to determine the number and disposition of the partial regions in such a manner as to obtain the maximum « sensibility », that is the maximum ease of distinction between the 5 distributions considered on the basis of the experimental results.

As regards the number of regions, it is to be noted that a high number has the advantage of allowing finer detail and so increases the sensibility; on the other hand, according to the number of events observed, there is the disadvantage of an increase in random fluctuations due to the smaller number of events in each region. Since both these effects have an influence which is roughly proportional to the square root of the number of regions, it can be assumed that they compensate each other. There remains however another circumstance which makes a fairly small number of regions preferable: that is the influence of errors of measurement. These in fact are transformed into an uncertainty in the position of the representative point; this uncertainty, if the point falls near the boundary between two regions, may make the attribution of the point to a definite region doubtful. It is clear that the higher the number of partial regions, the larger will be the number of points in such a position.

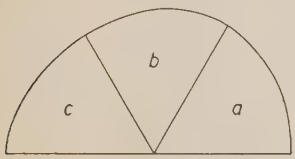

Fig. 2.

We therefore think it better to have a small number of regions. Fig. 2 shows the scheme of division adopted ([12]), which can be justified as follows: in the first place it upholds the symmetry of the problem, as the three regions a, b, c are equal; this symmetry has the physical result that the three regions have very simple characters. In fact $E_3 > E_1 > E_2$ for the points of area a; $E_1 > E_3 > E_2$ for those in b; and $E_1 > E_2 > E_3$ for those in c. It is therefore unnecessary to measure the energies in order to attribute an event to one of the three regions; all that is needed is the knowledge of the angles to establish which of the above three inequalities is satisfied (of course it is always necessary to know which is particle 3, which strictly requires that at least two tracks finish in emulsion, so that it is easy to measure their energies). The symmetry of the division adopted provides a second advantage; the explicit calculation of the probabilities is found to be much simpler than with a different division. In a non relativistic approximation the integrals which appear can be worked out exactly, and this fact has been used to verify that the resulting probabilities are sufficiently different in the various cases being studied. Since this condition was satisfied, the exact probabilities were then calculated numerically;

([12]) From the symmetry between the mesons 1 and 2 it is sufficient to use only half of Fig. 1.

the results are shown in the Table III, from which incidentally it can be seen that the relativistic corrections are not equally appreciable in all cases: in the case [1+], there are variations of ~ 0.1 in the probabilities of regions *a* and *c*, while the case [0 —] is completely unaltered; however, to have taken them into account seems quite justified.

TABLE III.

	Non-relativistic			Relativistic		
	a	*b*	*c*	*a*	*b*	*c*
[0 —]	0,333	0,334	0.333	0,333	0,334	0.333
[1 +]	0.425	0,334	0.241	0.521	0.334	0.145
[1 —]	0.196	0.608	0.196	0.211	0.579	0.210
[2 +]	0.241	0.334	0.425	0.182	0.334	0.484
[3 —]	0.294	0.412	0.294	0.292	0.417	0.291

With the latest data available at Rome (on 8 τ-mesons in which the π^- can be recognized) the frequencies are 0.25, 0.25, 0.50 for the regions *a*, *b*, *c* respectively. If it is correct to draw any conclusions from such a scanty amount of data, it seems that there are indications against the case [1+] and perhaps against [1 —]; we consider, however, that it will not be possible to draw any reliable conclusion until the number of τ-mesons in which the meson 3 is known has reached the order of a hundred.

Concluding I wish to thank Professors E. AMALDI and B. FERRETTI for their constant interest in this work; particular thanks are also due to Prof. B. F. TOUSCHEK for his notable contribution of ideas and for useful discussions.

RIASSUNTO

Si studia la distribuzione dei mesoni π nel decadimento del τ e si trovano espressioni approssimate per gli elementi di matrice. Si possono così giustificare i risultati di DALITZ, in cui vengono anche introdotte le correzioni relativistiche. Il confronto coll'esperienza è effettuato tenendo conto delle cariche dei π; tuttavia i dati sperimentali sono ancora troppo scarsi per dare una qualche informazione sulla parità e sullo spin del τ.

On Quantum Field Theory. II: Non-perturbative Equations and Methods.

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(ricevuto il 1° Marzo 1954)

Summary. — The evaluation of an element of the U-matrix between arbitrary initial and final states is reduced to that of a kernel, whose form depends only upon the number of particles involved and is given explicity as a perturbative expansion. Kernels are shown to satisfy systems of « branching equations », which hold independently of perturbation methods and can be taken as the axiomatic foundation of the theory, Lorentz covariance being manifest. Complete systems of such equations are given for the kernels and their derivatives with respect to the interaction strength λ: all other conceivable equations among kernels are necessarily deducible from them. All kernels corresponding to processes involving real bosons can be obtained, with simple integrations, from the kernels for purely fermionic processes; the branching equations for these are also explicity given and suffice to define the theory. A kernel, with its first and second λ-derivatives, satisfies a single integral relation. A variety of approximation methods are immediately deducible from the branching equations; they, while extending and generalizing the known ones, always permit, at least in principle, tests of convergence. Questions of renormalization, existence of solutions, etc., will be studied in the sequel to this paper.

1. – Introduction.

1·1. – The aim of this work, as well as of the preceding one, is to put the formal properties of the matrices S and U—whose evaluation is the central problem of a field theory—into a clear mathematical shape, which may permit a rigorous study of the extant fundamental questions: existence of solutions of the field equations, convergence of perturbation expansions, reliability of methods of approximation, and the like. It is indeed clear that, since in these

SOCIETÀ ITALIANA DI FISICA

SCUOLA INTERNAZIONALE DI FISICA

Sotto gli auspici del Ministero della Pubblica Istruzione
e del Consiglio Nazionale delle Ricerche

Corso estivo a Varenna, sul Lago di Como

18 Luglio - 7 Agosto 1954

1) Quest'anno, 1954, con il contributo del Ministero della Pubblica Istruzione, del Consiglio Nazionale delle Ricerche, dell'Università di Milano, delle Autorità e di Enti, Società e privati della Provincia di Como, la Scuola terrà a Varenna, sul Lago di Como, nella Villa Monastero — messa gentilmente a disposizione, insieme con il parco annesso, dall'Ente Villa Monastero — il 2° *Corso estivo* nel quale verranno trattate, in modo organico, approfondito e critico, *questioni relative alla rivelazione delle particelle elementari, e alle loro interazioni, con particolare riguardo alle particelle artificialmente prodotte e accelerate.*

2) La Direzione del Corso è affidata al professore Giampietro Puppi, professore di Fisica Superiore all'Università di Padova.

3) Gli insegnamenti fondamentali saranno i seguenti:
— *Produzione, assorbimento e diffusione dei mesoni π da parte dei nucleoni,*
— *Mesoni pesanti e iperoni prodotti da raggi cosmici o dal cosmotrone,*
— *Problemi delle grandissime energie e origine dei raggi cosmici,*
e verranno affidati ai professori
E. Fermi, professore all'University of Chicago (U.S.A.),
B. Rossi, professore al Massachussetts Institute of Technology, Cambridge (U.S.A.).
Inoltre un gruppo di lezioni sulla
— *Fotoproduzione dei mesoni π e fotoreazioni nucleari di grande energia*
verrà affidato al professore
G. Bernardini, professore all'Università di Roma, attualmente alla University of Illinois, Urbana (U.S.A.),
e un altro gruppo di lezioni sulle
— *Macchine acceleratrici e relativi apparati fondamentali di misura,*
verrà affidato a vari specialisti di varie nazionalità, in collaborazione col Centro Europeo di Ricerche Nucleari (C.E.R.N.).
Infine seminari e conferenze su questioni connesse con quelle che costituiscono l'oggetto del Corso saranno tenuti da eminenti fisici di varie nazionalità.

4) Il Corso avrà la durata di 21 giorni: si inizierà alle ore 18 di Domenica 18 Luglio e si chiuderà la sera di Sabato 7 Agosto. Le lezioni avranno luogo la mattina, dalle 8,30 alle 11, di Lunedì, Martedì, Mercoledì, Giovedì, Venerdì e Sabato; i seminari, le conferenze e le discussioni avranno luogo il pomeriggio, dalle 17,30 alle 19,30 degli stessi giorni eccetto il Sabato.
Le lezioni, i seminari, le conferenze e le discussioni saranno di regola tenuti in inglese o in francese.
È fatto obbligo agli allievi di frequentare regolarmente il Corso in tutta la sua attività.
Durante il Corso, una gita sarà organizzata dall'Ente Provinciale per il Turismo di Como.

5) Il numero totale degli allievi sarà di 30.
Chi desidera frequentare il Corso, dovrà entro il 5 Giugno 1954 farne pervenire domanda al

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Società Italiana di Fisica
Via Saldini, 50 MILANO (Italien)

fornendo, ben chiaramente, le seguenti informazioni:

a) Nome e Cognome.
b) Data e luogo di nascita.
c) Indirizzo attuale.
d) Titoli di studio universitari ottenuti e presso quale Università.
e) Professione attuale.
f) Elenco bibliografico dei lavori di Fisica già pubblicati.
g) Quale conoscenza orale e scritta ha dell'inglese e del francese.
h) Se verrà a Varenna da solo o accompagnato da familiari; nel caso che venga da solo, se è disposto a prendere alloggio in camera in comune con altro allievo; nel caso che venga accompagnato da familiari, quale è il grado di parentela di questi.

Inoltre, il richiedente deve allegare alla domanda una lettera di presentazione, rilasciata da un professore universitario di Fisica, il quale assicuri che il richiedente ha interesse per i suoi studi a frequentare il Corso e che possiede già adeguata preparazione.

L'accettazione delle domande è decisa dalla Presidenza della Società unitamente alla Direzione del Corso tenendo conto dei documenti presentati dai singoli richiedenti e di una equa ripartizione dei posti tra le varie nazioni cui questi appartengono.

Le comunicazioni delle deliberazioni saranno inviate agli interessati entro il 12 Giugno 1954.

6) Gli allievi saranno alloggiati o nella Foresteria della Villa Monastero o in alberghi di Varenna, vicino alla Villa, in camere da 1 o 2 letti.
Nella stessa Villa sarà organizzata la mensa della Scuola.

7) La spesa totale che ogni allievo deve sostenere per la frequenza al Corso, alloggio e vitto alla mensa della Scuola, è limitata, grazie alle sovvenzioni di cui la Scuola dispone, a L. 30 000, se egli prende alloggio in camera da solo, e a L. 25 000, se prende alloggio in camera con altro allievo o con familiari. Dette somme sono da versarsi, non più tardi del 22 Luglio, all'Amministrazione della Scuola, a Varenna, in moneta italiana.
Borse di studio, in numero limitatissimo, potranno essere accordate a quegli allievi che ne giustifichino il bisogno.

8) Per gli allievi che desiderino portare con loro i familiari, la Scuola, su richiesta che l'allievo dovrà unire alla domanda per frequentare il Corso, cercherà, entro le possibilità locali (che sono però assai limitate causa la stagione balneare), di trovare una sistemazione adeguata in alberghi di Varenna.
I familiari possono essere accolti, insieme con l'allievo, alla mensa della Scuola.
Tutte le spese per alloggio, vitto, ecc., relative ai familiari sono da computarsi a parte e totalmente a carico dell'allievo: esse, per trattamento uguale a quello fatto all'allievo, si aggirano intorno a L. 2 350 per giorno e per persona adulta, e dovranno essere regolate direttamente dall'allievo con l'albergatore.

9) Per ogni informazione relativa al Corso rivolgersi direttamente al Direttore di questo.

Milano, 31 Marzo 1954

Il Segretario della S.I.F.: G. C. Dalla Noce *Il Direttore del Corso*: G. Puppi

Il Presidente della S.I.F.: G. Polvani

questions physical intuition gives no hints at all, or perhaps hints to the contrary, all conclusions reached on grounds other than strict mathematics are forever objectionable.

The main hindrance to a formulation of the theory suitable for exact quantitative discussions has been, till now, the continual appearence of operators at all stages of approximation, whatever the method employed. Although these operators eventually disappear, their elimination with the customary technique leads soon to an inextricable algebraic complexity, which masks just those features of symmetry and compactness—always present when a theory is given its appropriate mathematical formulation—which should suggest the way to follow in a rigorous treatment. The first thing in such a program is, therefore, to eliminate the field operators from the very start, so to get formally simple expressions whose study—at least, until the problem of renormalisation comes into play—becomes then a *purely mathematical question, since all that can come from physics is already taken care of when removing the operators.*

We have shown in I ([1]) how this can be done, quite generally, for the perturbation expansions of elements of the S or U matrices. There we have examined the case of electrodynamics—whose consideration will suffice here too, the extension to other theories being trivially evident ([1])—under the restriction, however, that all states involved were taken orthogonal to one another. This limitation—quite expedient at the beginning, because avoiding complications which might have impaired a foresight of the properties to be looked for—is removed here, by the algebraic work of Appendices I and II. There we report concisely the rather toilsome developments necessary to get the wanted generalization, which is essential for our purposes and is taken as the starting point in the text.

1·2. – We consider throughout elements of the matrix $U(t_1, t_0)$ (starting from the language of the interaction representation and in configuration space); this, while avoiding us all the problems connected with the handling of non-convergent expressions (such as the adiabatic switching on and off of the interaction) which must be given due consideration in a rigorous treatment ([2]), gives the desired generality to our results, as no restriction to the energy shell is thereby implied.

Our criterion in investigating the properties of the U-matrix elements is

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([1]) E. R. Caianiello: *Nuovo Cimento*, **10**, 1634 (1953) (referred to as I in the text). See also: E. R. Caianiello and S. Fubini: *Nuovo Cimento*, **9**, 1218 (1952) (referred to as A in App. I).

([2]) B. Ferretti: *Nuovo Cimento*, **8**, 108 (1951).

the following. We have at hand the formal power-series expansion of an arbitrary element; suppose, however, that such an expansion is *quantitatively* meaningless, for some or even all the values of the interaction strength; if a solution exists at all, whichever may be our method of attaining it, some of its *qualitative*, formal properties are certainly the same as those of the perturbation expansion—provided, of course, it can be *formally* reduced to such expansion. The Fredholm theory is an outstanding, but by no means unique, example of the sort. We endeavor, therefore, to obtain complete statements of properties of this nature: among them, the integral and integro-differential equations derived in Part **4**.

Before attacking this question it is convenient, although not necessary, to take another step. An element of the U-matrix depends upon two conceptually different data: the specific particle states between which the element is taken, and the number of particles involved. It is only the second which, as is intuitive and will be shown in detail, determines its relevant formal properties. It appears therefore desirable to be able to express, to within a well-determined numerical factor, a matrix element by means of a « kernel », whose form depends only upon the number of particles involved, and of the Fock-space wave functions of the initial and final states, duly normalized and symmetrized.

This question is studied in Part **2**. It comes out that, as is expected, the handling of kernels—whose perturbation expansion is deduced—is much simpler than that of matrix elements; in particular, the kernels defined by us include already all the effects of symmetrizations and antisymmetrizations. All relations among kernels can, at any stage, be changed into relations among matrix elements, if one wishes so.

All numerical coefficients and signs come out automatically at the right places, with the right values; we have given some care to this point, which may save needless head-aches in calculating. As a check on the numerical part of our work, and also to show the consistency of the formalism adopted, we dedicate Part **3** to a brief digression, in which the two particular cases arising when only one of the fields is quantized are considered: the well-known results of FEYNMAN ([3]) and GLAUBER ([4]) are seen to follow with great ease, and without any physical thinking, from the sheer mechanism of mathematics, as was in our program.

([3]) R. P. FEYNMAN: *Phys. Rev.*, **78**, 749 (1949).

([4]) R. J. GLAUBER: *Phys. Rev.*, **84**, 395 (1951). Similar results have been obtained also by other authors; (ref. 8; also: W. THIRRING and B. F. TOUSCHEK: *Phil. Mag.*, **42**, 244 (1951)); we refer here to GLAUBER's work because his notation comes closer to ours.

1·3. – The main results of this work are deduced in Part 4. The power-series expansion of a kernel is strongly reminiscent of the expansions familiar in the Fredholm theory (it is exactly so, if only the fermion field is quantized); we are, therefore, in the position of one who, knowing the resolving kernel, proposes to deduce the equation. Things are, however, complicated by the fact that, in a two-field theory, two different propagation functions appear: D^F and S^F in our case; this leads to the presence of both hafnians and determinants in the expansion. One expects therefore to get an infinite number of integral equations, which connect kernels corresponding to different particle numbers; and other such equations, involving also the derivatives of the kernels with respect to the interaction strength.

We find « complete systems » of equations of this sort, which it seems appropriate to call « branching equations »; by « complete » we mean that any other possible equation of the « branching type » among kernels derives from those of the systems, by iteration. Since, conversely, it is not hard to see that the perturbation expansions can be deduced unambiguously from the branching equations, and since the latter can be derived formally from the field equations without having to pass through the perturbation expansions, the conclusion is that the solutions of these equations, *if any*, are the correct physical ones even if the perturbation expansions happen to be meaningless.

This suggests an interesting possibility; namely, that a field theory may be formulated starting directly from equations of this sort for the kernels, Lorentz covariance being thus manifest; no mention needs be made of Hamiltonians, and one would have a theory staying in between the Hamiltonian and the S-matrix view-points, with all the advantage of the first. This is no novelty in principle, except that we give explicit formulation to all the possible branching equations, in the most general case.

The results just mentioned, as we have said, might have been expected to hold on quite general grounds. Not so, however, for some of their consequences, which seem rather remarkable. Such is the fact that all the kernels corresponding to processes in which arbitrary numbers of bosons occur can be derived, by means of simple integrations, from the kernels corresponding to processes involving only fermions: this reduces greatly the complexity of the question, since one has then to study only the latter, whose defining equations are rather simple formally. Also, a single « purely fermionic kernel » can be shown to satisfy an integro-differential equation: integral with respect to the space-time variables, differential with respect to the square of the interaction strength, of which the first and second derivatives enter the equation.

It is clearly possible to derive many more consequences of a similar nature, as well as relations among the kernels on other grounds (for instance, from the unitarity condition); it seems appropriate, however, to differ a more detailed study to a later time, when, having already studied the renormalization

problem, it may be possible to decide which of them are actually useful for our purpose.

Appendix III contains an algebraic proof of Furry's theorem for kernels; although unnecessary as such, it serves to establish some properties stated in the text, which are not mentioned here for brevity.

1·4. – The very form of the branching equations shows that it is possible to approximate their solutions in a great variety of ways. In Part 5 we outline briefly one possible method, which appears to be a combination of the perturbation and of the TAMM-DANCOFF [5] view-points, uniting to the advantages of the second a greater simplicity—no operators, and only one type of integral equation to be solved once for all—with the additional possibility of evaluating the error committed at each step. Although we do not enter here into this question, its renormalization with the available techniques seems feasible.

Our treatment of this more technical side, is, allegedly, very incomplete. It is very possible that some other of the methods of solution, which can be derived on inspection from the branching equations, may prove more convenient than the one we mention here. Before making any statement on such questions, that is, on « how » to approximate, one should know « what » he is trying to approximate. This requires a previous study of renormalization; we have no doubt as to its feasibility, the problem seems rather that of obtaining an « efficient » technique. To it we propose to dedicate our next work on the argument. We may just mention here that there are indications that, if nothing else, the branching equations appear as a useful tool in the study of the renormalization and convergence of the perturbation expansion.

2. – Passage to Fock-space.

2·1. – We adopt throughout the definitions and notation of I, but for a few trivial changes indicated below. We propose in this part to express the element $M_{FI}(t_1, t_0)$ of the matrix $U(t_1, t_0)$—taken between an initial state with a photons, n electrons, and m positrons and a final state with b photons, p electrons and q positrons—by means of the wave functions of these states (with the due properties of symmetry and normalization) and of a « kernel », whose form depends only upon the numbers $a+b=P_0$ and $n+q=m+p=N_0$,

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([5]) I. TAMM: *Journ. Phys. Rev.*, **78**, 382 (1950). See, however: G. MORPURGO and B. F. TOUSCHEK: *Nuovo Cimento*, **10**, 1681 (1953).

but not upon the particular states in which the particles are taken initially and finally. It is required that $M_{FI}(t_1, t_0)$ be obtained, to within a numerical coefficient, by means of well defined operations upon the product of these wave functions and of the kernel, such as integrations, averages, or limiting processes. Since it is clear that the only quantities endowed with physical significance are the matrix elements, it will be appropriate to regard kernels, and what operations may be devised to obtain the matrix elements from them, as purely mathematical artifices, which may be substituted, indifferently, with others more appropriate to particular problems, or suppressed altogether if convenient: every relation among kernels being immediately translatable into a relation among matrix elements. The advantage of working with kernels rather than with matrix elements lies, as far as the present work is concerned, in the fact that formal work is rendered thereby much more perspicuous and compact, and that structural relations among kernels are common to all the infinite matrix elements with the same P_0 and N_0; the kernels exhibit most clearly the symmetries we are interested in and appear to be, so to say, the « backbone » of the matrix elements. Their role in the present theory is analogous to, and inspired by, that of Fredholm's resolving kernels and minors.

The main result of I, formula (I.34), was deduced under the two following restrictive conditions: that no two initial, or final, bosons should be in the same state, and that no (boson or fermion) initial state should coincide with some final state. In Appendices I and II we report the algebraic steps which remove both restrictions; the result is formula (2) below, which reduces to (I.34) if these restrictive conditions happen to be valid, and is the starting point of our considerations.

As in I, it is possible to consider separately the electro-magnetic and the electron-positron contributions. The state vectors which describe the initial and final situations of the electro-magnetic field, with correct normalization, are now, denoting with $|0\rangle$ the interaction-representation vacuum:

$$(1)\qquad \begin{cases} |\Psi_I^{(b)}\rangle = |\Psi(\tau_1, \tau_2, \ldots, \tau_\alpha)\rangle = \dfrac{1}{\sqrt{\tau_1!\,\tau_2!\ldots\tau_\alpha!}}\, a_\alpha^*\ldots a_\alpha^*\ldots a_1^*\ldots a_1^*\,|0\rangle\,, \\ |\Psi_F^{(b)}\rangle = |\Psi(\sigma_{1'}, \sigma_{2'}, \ldots, \sigma_{\beta'})\rangle = \dfrac{1}{\sqrt{\sigma_{1'}!\,\sigma_{2'}!\ldots\sigma_{\beta'}!}}\, a_{\beta'}^*\ldots a_{\beta'}^*\ldots a_{1'}^*\ldots a_{1'}^*\,|0\rangle\,, \end{cases}$$

where $\tau_1+\tau_2+\ldots+\tau_\alpha=a$, $\sigma_{1'}+\sigma_{2'}+\ldots+\sigma_{\beta'}=b$, and the first of (1), say, denotes that there are initially τ_1 photons in state 1, τ_2 in state 2, ..., τ_α in state α. Furthermore, initial (unprimed) states may coincide at will with final (primed) states; these are the differences between (1) and (I.8). The initial and final situations of the fermion field are described by (I.17), again including, now, the possibility that initial and final states coincide.

The wanted generalization of (I.34) is, then:

$$(2)\qquad M_{FI}(t_1, t_0) = \sum_N M_{FI}^{(N)}(t_1, t_0) =$$

$$= (-1)^{p_{FI}} \frac{1}{\sqrt{\sigma_{1'}!\,\sigma_{2'}!\ldots\sigma_{\beta'}!\,\tau_1!\,\tau_2!\ldots\tau_\alpha!}} \sum_{N(P_0)} \frac{\lambda^N}{N!} \int \sum \gamma'^{\mu_1}_{\alpha_1\beta_1} \ldots \gamma'^{\mu_N}_{\alpha_N\beta_N} [\xi_1\xi_2\ldots\xi_N z^{(1)} z^{(2)} \ldots z^{(P_0)}$$

$$\cdot \begin{pmatrix} \xi_1\xi_2\ldots\xi_N v^{(1)} v^{(2)} \ldots v^{(N_0)} \\ \xi_1\xi_2\ldots\xi_N u^{(1)} u^{(2)} \ldots u^{(N_0)} \end{pmatrix},$$

where: λ is the electric charge (in units $\hbar = c = 1$); $\sum_{N(P_0)}$ means summation over *all* the values of N having the same parity as P_0; $p_{FI} = p(m+n) + \binom{n}{2} - \binom{p}{2}$ is the same phase factor found in I; $\int \sum$ means summations over all component indices and integration over all the 4-dimensional variables $\xi_1, \xi_2, \ldots, \xi_N$, performed over the same volume V (eventually, all space) and time interval t_0, t_1 for all variables (we denote throughout the dummy integration variables $1, 2, \ldots, N$ of I with $\xi_1, \xi_2, \ldots, \xi_N$);

the prime is affixed to the determinants in order to adhere to the notation established in I (that is, we find it convenient to write the γ'^{μ}'s out of the determinants); it will be dropped henceforth, as no confusion may arise;

hafnians are expanded with the rules:

$$(3)\qquad \begin{cases} [\xi_h\xi_k] = [\xi_k\xi_h] = \frac{1}{2}\,\delta_{\mu_h\mu_k} D^F(\xi_h, \xi_k)\,, \\ [z^{(h)}\xi_h] = [\xi_h z^{(h)}] = z^{(h)}_{\mu_h}(\xi_h)\,, \end{cases}$$

$$(4)\qquad [z^{(h)} z^{(k)}] = [z^{(k)} z^{(h)}] = \begin{cases} 0 & \text{for } h, k \leqslant a \text{ (initial states)} \\ 0 & \text{» } h, k > a \text{ (final states)} \\ \bar{\delta}_{hk} & \text{» } h \leqslant a,\ k > a, \text{ or } k \leqslant a,\ h > a \text{ (one initial and one final state)} \end{cases}$$

$$z^{(a+1)} = z^{(1')*};\quad \ldots;\quad z^{(P_0)} = z^{(b')*}$$

(a star will denote complex conjugate or hermitian conjugate, as will be the case; the symbol $\bar{\delta}_{hk}$ is used here, and in the following, to denote 1 if the initial (final) state labelled with h coincides with the final (initial) state labelled with k, and 0 otherwise);

determinants are expanded with the rules (dropping primes; see,

also, 22)):

$$
(5)\qquad \left\{\begin{array}{l}
(\xi_h\xi_k) = \dfrac{1}{2}\, S^F_{\beta_h\alpha_k}(\xi_h, \xi_k) \\
(u^{(k)}\xi_h) = (\xi_h u^{(k)}) = u^{(k)}_{\beta_h}(\xi_h) \\
(v^{(k)}\xi_h) = (\xi_h v^{(k)}) = v^{(k)}_{\alpha_h}(\xi_h) \\
u^{(u+1)} = \bar{v}^{(1')}, \ldots, u^{(N_0)} = \bar{v}^{(q')};\ v^{(m+1)} = \bar{u}^{(1')}, \ldots, v^{(N_0)} = \bar{u}^{(P')},
\end{array}\right.
$$

$$
(6)\qquad (u^{(h)}v^{(k)}) = (v^{(k)}u^{(h)}) = \left\{\begin{array}{ll}
0 & \text{if } h \leqslant n,\ k \leqslant m, \text{ or } h > n,\ k > m, \\
-\bar{\delta}^{\mathrm{el}}_{hk} & \text{if } h > n,\ k \leqslant m, \\
+\bar{\delta}^{\mathrm{pos}}_{hk} & \text{if } h \leqslant n,\ k > m .
\end{array}\right.
$$

($\bar{\delta}^{\mathrm{el}}$ denotes that h and k are labels of electron states, $\bar{\delta}^{\mathrm{pos}}$ same for positron states).

It appears that the only difference between (I.34) and (2) is—aside from the numerical coefficient of (2) and the greater range of $\sum_{N(P_0)}$—that contained in the definitions of (4) and (6) (which in I are always $\equiv 0$).

Before going any further, it will be convenient to dedicate the next section to a short *résumé* of well-known formulae which are used later, and to a clear settling of trivial matters like signs and normalization coefficients, which are however of relevant practical value.

2·2. – Unless otherwise stated, all results derived in this work hold also if external fields are present; this is *per se* shown by our use of the functions $D^F(\xi_h, \xi_k)$ and $S^F(\xi_h, \xi_k)$. The validity of our considerations is in no way restricted to the case that initial and final states be described by plane waves; the introduction of kernels, however, is facilitated if we assume that the initial and final states are eigenstates of the unperturbed Hamiltonians; this we do here. Should one wish to eliminate this restriction, a re-examination might prove necessary of the artifice used in the following to connect matrix elements with kernels; we deem unnecessary a more detailed study of this question, since, more simply, it is sufficient to re-express, at the end, all the results obtained for kernels in terms of matrix elements to regain the wanted generality.

a) *Electromagnetic field.* – It is expanded in normal modes as:

$$
(7)\qquad A_\mu(x) = \sum_\varrho \left(z^{(\varrho)}_\mu(x) a_\varrho + z^{(\varrho)*}_\mu(x) a^*_\varrho\right) .
$$

In the particular case of plane waves one has:

$$A_\mu(x) = \sum_{\boldsymbol{k}} \sum_{\lambda=1}^{4} \left[\cos(\lambda(k), \mu) \frac{e^{ikx}}{\sqrt{2k_0 V}} a_{\boldsymbol{k},\lambda} + \cos(\lambda(k), \mu) \frac{e^{-ikx}}{\sqrt{2k_0 V}} a^*_{\boldsymbol{k},\lambda} \right],$$

where $\varepsilon^\lambda_\mu(k) = \cos(\lambda(k), \mu)$ are the unit vectors of the frames associated with each $\boldsymbol{k}$, and $\varrho \equiv \boldsymbol{k}, \lambda$, $a_\varrho = a_{\boldsymbol{k},\lambda}$. Here:

$$z^{(\varrho)}_\mu(x) = \cos(\lambda(k), \mu) \frac{e^{ikx}}{\sqrt{2k_0 V}}.$$

In general:

$$z^{(\varrho)}_\mu(x) = \frac{1}{\sqrt{2E_\varrho}} \zeta^{(\varrho)}_\mu(x), \tag{8}$$

where the $\zeta^{(\varrho)}_\mu(x)$ are a complete set of orthonormal eigenfunctions of the energy. As is well known, from the definition of $[\xi_h \xi_k]$ as a vacuum expectation value one has:

$$[\xi_h \xi_k] = \frac{1}{2} \delta_{\mu_h \mu_k} D^F(\xi_h, \xi_k) = \begin{cases} \sum_\varrho z^{(\varrho)}_{\mu_h}(\xi_h) z^{(\varrho)*}_{\mu_k}(\xi_k) & \text{for} \quad \xi^0_h > \xi^0_k, \\ \sum_\varrho z^{(\varrho)}_{\mu_k}(\xi_k) z^{(\varrho)*}_{\mu_h}(\xi_h) & \text{for} \quad \xi^0_h < \xi^0_k. \end{cases} \tag{9}$$

b) *Electron – positron field.* – We decompose ψ and $\bar{\psi}$ in the following manner:

$$\psi(x) = u(x) + \bar{v}(x); \qquad \bar{\psi}(x) = \bar{u}(x) + v(x), \tag{10}$$

with

$$\begin{cases} u(x) = \sum_\varrho u^{(\varrho)}(x) a_\varrho; & \bar{v}(x) = \sum_\varrho \bar{v}^{(\varrho)}(x) b^*_\varrho, \\ \bar{u}(x) = \sum_\varrho \bar{u}^{(\varrho)}(x) a^*_\varrho; & v(x) = \sum_\varrho v^{(\varrho)}(x) b_\varrho, \end{cases} \tag{11}$$

where:

$$\bar{u} = u^* \gamma^4, \quad \bar{v} = \gamma^4 v^*; \qquad \int \mathrm{d}\boldsymbol{x}\, u^{(r)*}(x) u^{(s)}(x) = \int \mathrm{d}\boldsymbol{x}\, v^{(r)*}(x) v^{(s)}(x) = \delta_{rs},$$

and $u^{(\varrho)}$ is the wave function of an electron, $v^{(\varrho)*}$ the wave function of the negative-energy electron having momentum and energy opposite to that of the positron. (In the Majorana representation, v is the wave function of the

positron itself, since then: $C \equiv 1$, $u^c \equiv u^*$. It follows then from the Dirac equation that:

$$(12) \qquad \begin{cases} \sum_{\lambda=1}^{4} \int \mathrm{d}\boldsymbol{x} \bar{u}_\lambda^{(r)}(x) u_\lambda^{(s)}(x) = \dfrac{m}{E_r} \delta_{rs} , \\ \sum_{\lambda=1}^{4} \int \mathrm{d}\boldsymbol{x}\, v_\lambda^{(r')}(x) \bar{v}_\lambda^{(s')}(x) = -\dfrac{m}{E_{r'}} \delta_{r's'} , \end{cases}$$

the energies E_r (of an electron) and $E_{r'}$ (of a positron) being always positive. As was the case with formula (9), we have now:

$$(13) \qquad (\xi_h \xi_k) = \frac{1}{2} S^F_{\beta_h \alpha_k}(\xi_h, \xi_k) = \begin{cases} -\sum_{\varrho} u^{(\varrho)}_{\beta_h}(\xi_h) \bar{u}^{(\varrho)}_{\alpha_k}(\xi_k) & \text{for} \quad \xi_h^0 > \xi_k^0 \\ +\sum_{\varrho} v^{(\varrho)}_{\alpha_k}(\xi_k) \bar{v}^{(\varrho)}_{\beta_h}(\xi_h) & \text{for} \quad \xi_h^0 < \xi_k^0 . \end{cases}$$

2·3. – We have already emphasized that kernels, although powerful as mathematical tools, are not, by themselves, the physical entities; we cannot therefore, exclude that other equivalent expressions may be given for them. Their introduction, if not imposed by physics, is strongly suggested by the peculiar nature of formula (2), and easily achieved by guesswork. Our choice, made with the criterion of maximum simplicity and symmetry, is expressed by formula (30). The problem is, then, to verify its correctness and to put signs and coefficients straight. We would expect the following to happen: a kernel, as defined by (30), is a function of as many space-time variables, and spinor or vector indices, as there are particles in both initial and final states (the fact that, thus far, we can define kernels only by means of power-series expansions is inessential in this respect); multiplication of the kernel by the wave functions of the initial and final states, suitably written with the same variables and indices, plus space-integrations and summations with respect to these, should yield, to within a numerical coefficient, the matrix element. This actually always happens, if formula (I.34) can be used. If, however, we must use formula (2), there may be cases in which places in the determinants or hafnians which should, correctly, be occupied by zeros, come out instead filled with time-dependent phase factors. Although this may not be so bad, from what we know of these matters, it is certainly desirable to give rules whereby formula (2) may be obtained correctly in any case. *One* possible way out is to introduce suitable time-averages, as defined below. (We refrain from formulating the simple physical interpretation of this procedure, for which we claim no novelty, and of many other steps taken elsewhere, because of its... utter irrelevance to physics). Our prescription is the following:

Integrate upon the space parts of the arguments of all wave functions,

taken at times $< t_0$ for the initial and $> t_1$ for the final particles; then, average over times from $-\infty$ to t_0 for the first, from t_1 to $+\infty$ for the second (and sum over all spinor and vector indices). To save writing, all these operations will be indicated with the symbol $\hat{\int}$. Thus, for an initial particle:

$$\hat{\int} \dots w_{\lambda_h}(x_h) = \sum_{\lambda_h=1}^{4} \lim_{T_h \to \infty} \frac{1}{t_0 + T_h} \int_{-T_h}^{t_0} \mathrm{d}x_h^0 \int \mathrm{d}\boldsymbol{x}_h \dots w_{\lambda_h}(x_h) ,$$

and for a final particle:

$$\hat{\int} \dots \overline{w}_{\lambda_h}(x_h) = \sum_{\lambda_h=1}^{4} \lim_{T_h \to \infty} \frac{1}{T_h - t_1} \int_{t_1}^{T_h} \mathrm{d}x_h^0 \int \mathrm{d}\boldsymbol{x}_h \dots \overline{w}_{\lambda_h}(x_h) .$$

We proceed now to show how this prescription is effective in re-expressing (2) in Fock space. It is convenient to start from the separate consideration of the hafnian and of the determinant in the term $M_{FI}^{(N)}(t_1, t_0)$ of (2), so to derive the expressions (30) of the kernels together with the appropriate coefficients (32).

a) *Hafnian.* – We denote the indices characterizing the vector components of the wave functions of initial or final states (same, later, for spinor components) with primed Greek letters, to distinguish them from the unprimed dummy indices; let the space-time arguments of the boson wave functions be denoted with $t_1, t_2, \dots, t_{P_0}$ (not to be confused with the limits t_0 and t_1 of the time integrations!). We show first that:

$$(14) \qquad [\xi_1 \xi_2 \dots \xi_N z^{(1)} z^{(2)} \dots z^{(a)} z^{(1')*} z^{(2')*} \dots z^{(b')*}] =$$

$$= 2^{P_0/2} \sqrt{[E]_b} \hat{\int} [\xi_1 \xi_2 \dots \xi_N t_1 t_2 \dots t_{P_0}] \cdot \zeta_{\mu_1'}^{(1)}(t_1) \dots \zeta_{\mu_a'}^{(a)}(t_a) \cdot \zeta_{\mu_{a+1}'}^{(1')*}(t_{a-1}) \dots \zeta_{\mu_{P_0}'}^{(b')*}(t_{P_0}) ,$$

with $[E]_b$ product of all the energies of the P_0 photons. In (14), initial and final particles are explicitly indicated; the hafnian at right is expanded by setting

$$(15) \qquad [\xi_h t_k] = \frac{1}{2} \delta_{\mu_h \mu_k'} D^F(\xi_h, t_k) ,$$

etc., and likewise, later, for the symbols $(\xi_h y_k)$, etc..

The proof of (14) is quite simple. From (9) we have

$$(16) \qquad \begin{cases} \hat{\int} [\xi_h t_k] \zeta_{\mu_k'}^{(r)}(t_k) = \dfrac{1}{\sqrt{2E_r}} z_{\mu_h}^{(r)}(\xi_h) , \\ \hat{\int} [\xi_h t_k] \zeta_{\mu_k'}^{(r')*}(t_k) = \dfrac{1}{\sqrt{2E_r}} z_{\mu_h}^{(r')*}(\xi_h) . \end{cases}$$

For the cross terms between initial and final states, as defined by (4), we find:

$$\int^{\wedge} [t_h t_k] \zeta^{(r)}_{\mu'_h}(t_h) \zeta^{(s')*}_{\mu'_k}(t_k) = \frac{1}{\sqrt{2E_r}\cdot\sqrt{2E_{s'}}} \cdot \bar{\delta}_{rs'} \,. \tag{17}$$

In all these cases the time average is ineffective: it acts like the identity. It is easily seen, however, that it annihilates all cross terms between initial (final) states which may happen to be $\neq 0$ after the space integration: this is our reason for using energy eigenfunctions. (The explicit verification of (16) and (17), made for instance in the case of no external fields, by using plane waves and the analytic expression of the functions $D^F(x_h - x_k)$, is perhaps instructive, and quite simple). The numerical factors appearing in (16) and (17) account for the coefficient in (14).

From the properties of hafnians and permanents stated in I, and from obvious symmetry considerations, it also follows that:

$$\left[\xi_1\xi_2 \dots \xi_N z^{(1)} z^{(2)} \dots z^{(P_0)}\right] = 2^{P_0/2}\sqrt{[E]_b}\,\frac{1}{a!\,b!} \cdot \tag{18}$$

$$\cdot \int^{\wedge} \left[\xi_1\xi_2 \dots \xi_N t_1 t_2 \dots t_{P_0}\right] \cdot \left[\zeta^{(1)}_{t_1}\zeta^{(2)}_{t_2} \dots \zeta^{(a)}_{t_a}\right] \cdot \left[\zeta^{(1')*}_{t_{a+1}}\zeta^{(2')*}_{t_{a+2}} \dots \zeta^{(b')*}_{t_{P_0}}\right] ,$$

where, in keeping with our previous notation, we set:

$$[\zeta^{(r)} t_h] = \zeta^{(r)}_{\mu'_h}(t_h) \; ; \qquad [\zeta^{(r)*}(t_h)] = \zeta^{(r')*}_{\mu'_h}(t_h) \,.$$

We have only to observe that the wave function of a system of photons corresponding to the situation described by (1), normalized to unity, is, for initial photons, a tensor with components:

$$\frac{1}{\sqrt{\tau_1!\,\tau_2! \dots \tau_\alpha!\,\sqrt{a!}}} \begin{bmatrix} \zeta^{(1)} & \zeta^{(2)} \dots & \zeta^{(a)} \\ t_1 & t_2 \dots & t_a \end{bmatrix} = \varphi_{\mu'_1 \cdot \mu'_2, \dots, \mu'_a}(t_1, t_2, \dots, t_a) = \varphi_I^{(b)} \,, \tag{19}$$

and, for final photons:

$$\frac{1}{\sqrt{\sigma_{1'}!\,\sigma_{2'}! \dots \sigma_{\beta'}!\,\sqrt{b!}}} \begin{bmatrix} \zeta^{(1')}\zeta^{(2')} \dots & \zeta^{(b')} \\ t_{a+1}\, t_{a+2} \dots & t_{P_0} \end{bmatrix} = \tag{20}$$

$$= \varphi_{\mu'_{a+1}, \mu'_{a+2}, \dots, \mu'_{P_0}}(t_{a+1}, t_{a+2}, \dots, t_{P_0}) = \varphi_F^{(b)} \,,$$

to conclude that the photonic contribution to $M^{(N)}_{FI}(t_1, t_0)$ can be written as:

$$\frac{1}{\sqrt{\sigma_{1'}!\sigma_{2'}! \dots \sigma_{\beta'}!\,\tau!\,\tau_2! \dots \tau_\alpha!}} \left[\xi_1\xi_2 \dots \xi_N z^{(1)} z^{(2)} \dots z^{(P_0)}\right] = \tag{21}$$

$$= \frac{2^{P_0/2}\sqrt{[E]_b}}{\sqrt{a!\,b!}} \int^{\wedge} \varphi_F^{(b)*} \left[\xi_1\xi_2 \dots \xi_N t_1 t_2 \dots t_{P_0}\right] \varphi_I^{(b)} \,,$$

which accomplishes the wanted transformation to Fock space in a quite symmetrical manner.

b) *Determinants.* – In the same manner, there is no trouble in verifying that

$$(22)\qquad \begin{pmatrix} \xi_1\xi_2 \dots \xi_N\ v^{(1)}v^{(2)}\dots v^{(N_0)} \\ \xi_1\xi_2 \dots \xi_N\ u^{(1)}u^{(2)}\dots u^{(N_0)} \end{pmatrix} =$$

$$= \begin{vmatrix} (\xi_1\xi_1) & \dots & (\xi_1\xi_N) & (\xi_1u^{(1)}) & \dots (\xi_1u^{(n)}) & (\xi_1\bar{v}^{(1')}) & \dots (\xi_1\bar{v}^{(q')}) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ (\xi_N\xi_1) & \dots & (\xi_N\xi_N) & (\xi_Nu^{(1)}) & \dots (\xi_Nu^{(n)}) & (\xi_N\bar{v}^{(1')}) & \dots (\xi_N\bar{v}^{(q')}) \\ (v^{(1)}\xi_1) & \dots & (v^{(1)}\xi_N) & 0 & \dots\ 0 & \bar{\delta}^{\mathrm{pos}}_{11'} & \dots\ \bar{\delta}^{\mathrm{pos}}_{1q'} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ (v^{(m)}\xi_1) & \dots & (v^{(m)}\xi_N) & 0 & \dots\ 0 & \bar{\delta}^{\mathrm{pos}}_{m1'} & \dots\ \bar{\delta}^{\mathrm{pos}}_{mq'} \\ (\bar{u}^{(1')}\xi_1) & \dots & (\bar{u}^{(1')}\xi_N) & -\bar{\delta}^{\mathrm{el}}_{1'1} & \dots -\bar{\delta}^{\mathrm{el}}_{1'n} & 0 & \dots\ 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ (u^{(p')}\xi_1) & \dots & (\bar{u}^{(p')}\xi_N) & -\bar{\delta}^{\mathrm{el}}_{p'1} & \dots -\bar{\delta}^{\mathrm{el}}_{p'n} & 0 & \dots\ 0 \end{vmatrix} =$$

$$= \frac{[E]_f}{m^{2N_0}} \int \begin{pmatrix} \xi_1\xi_2 \dots \xi_N x_1x_2 \dots x_{N_0} \\ \xi_1\xi_2 \dots \xi_N y_1y_2 \dots y_{N_0} \end{pmatrix} \cdot v^{(1)}_{\beta'_1}(x_1) \dots v^{(m)}_{\beta'_m}(x_m) \cdot \bar{u}^{(1')}_{\beta'_{m+1}}(x_{m+1}) \dots \bar{u}^{(p')}_{\beta'_{N_0}}(x_{N_0}) \cdot$$
$$\cdot u^{(1)}_{\alpha_1}(y_1) \dots u^{(n)}_{\alpha_n}(y_n) \cdot \bar{v}^{(1')}_{\alpha'_{n+1}}(y_{n+1}) \dots \bar{v}^{(q')}_{\alpha'_{N_0}}(y_{N_0}) ,$$

with: $[E]_f$ = product of all the (positive!) energies of the $2N_0$ fermions. From (13) we have indeed:

$$(23)\qquad \begin{cases} \int (x_k\xi_h) v^{(r)}_{\beta_k}(x_k) = -\dfrac{m}{E_r} v^{(r)}_{\alpha_h}(\xi_h) ; \qquad \hat{\int} (\xi_h y_k) u^{(r)}_{\alpha'_k}(y_k) = -\dfrac{m}{E_r} u^{(r)}_{\beta_h}(\xi_h) , \\ \hat{\int} (x_k\xi_h) \bar{u}^{(s')}_{\beta'_k} u(x_k) = -\dfrac{m}{E_{s'}} \bar{u}^{(s')}_{\alpha_h}(\xi_h) ; \qquad \hat{\int} (\xi_h y_k) \bar{v}^{(s')}_{\alpha'_k}(y_k) = -\dfrac{m}{E_{s'}} \bar{v}^{(s')}_{\beta_h}(\xi_h) . \end{cases}$$

All cross terms between u and v, u and v, if not already $=0$, become so because of the time average; for the cross terms between v and v, $\bar{u}$ and u, we find from (12), respectively:

$$(24)\qquad \frac{m}{E_r} \int v^{(r)}_{\alpha_k}(\xi_k) \bar{v}^{(s')}_{\alpha'_k}(\xi_k) = \frac{m^2}{E_rE_{s'}} \bar{\delta}^{\mathrm{pos}}_{rs'} ; \quad -\frac{m}{E_r} \hat{\int} \bar{u}^{(s')}_{\beta'_h}(\xi_h) u^{(r)}_{\beta_h}(\xi_h) = \frac{m^2}{E_rE_{s'}} \bar{\delta}^{\mathrm{el}}_{rs'} .$$

The numerical coefficient in (22) is thus accounted for by those appearing in (23) and (24); on changing the sign of the last N_0 rows and columns, the determinant obtained with these operations becomes identical with the one at left in (22). Finally, as before, we can write:

$$(25)\qquad \begin{pmatrix} \xi_1\xi_2 \dots \xi_N v^{(1)} v^{(2)} \dots v^{(N_0)} \\ \xi_1\xi_2 \dots \xi_N u^{(1)} u^{(2)} \dots u^{(N_0)} \end{pmatrix} =$$

$$= \frac{[E]_f}{m^{2N_0}} \frac{1}{\sqrt{m!\, n!\, p!\, q!}} \int \bar{\psi}_F^{\text{pos}}\, \bar{\psi}_F^{\text{el}} \begin{pmatrix} \xi_1\xi_2 \dots \xi_N x_1 x_2 \dots x_{N_0} \\ \xi_1\xi_2 \dots \xi_N y_1 y_2 \dots y_{N_0} \end{pmatrix} \cdot \psi_I^{\text{pos}} \psi_I^{\text{el}} \,,$$

where the wave functions of initial and final electrons and positrons, normalized to unity, are given by:

$$(26)\qquad \psi_I^{\text{el}} = \psi_{\alpha_1', \alpha_2', \dots, \alpha_n'}(y_1, y_2, \dots, y_n) = \frac{1}{\sqrt{n!}} \begin{pmatrix} u^{(1)} u^{(2)} \dots u^{(n)} \\ y_1 \quad y_2 \quad \dots y_n \end{pmatrix};$$

$$(u^{(k)} y_h) = u_{\alpha_h'}^{(k)}(y_h) \,,$$

$$(27)\qquad \psi_I^{\text{pos}} = \psi_{\beta_1', \beta_2', \dots, \beta_m'}(x_1, x_2, \dots, x_m) = \frac{1}{\sqrt{m!}} \begin{pmatrix} v^{(1)} v^{(2)} \dots v^{(m)} \\ x_1 \; x_2 \; \dots x_m \end{pmatrix};$$

$$(v^{(k)} x_h) = v_{\beta_h'}^{(k)}(x_h) \,,$$

$$(28)\qquad \psi_F^{\text{el}} = \psi_{\beta'_{m+1}, \beta'_{m+2}, \dots, \beta'_{N_0}}(x_{m+1}, x_{m+2}, \dots, x_{N_0}) = \frac{1}{\sqrt{p!}} \begin{pmatrix} u^{(1')} u^{(2')} \dots u^{(p')} \\ x_{m+1} x_{m+2} \dots x_{N_0} \end{pmatrix};$$

$$(u^{(k')} x_h) = u_{\beta_h'}^{(k')}(x_h) \,,$$

$$(29)\qquad \psi_F^{\text{pos}} = \psi_{\alpha'_{n+1}, \alpha'_{n+2}, \dots, \alpha'_{N_0}}(y_{n+1}, y_{n+2}, \dots, y_{N_0}) = \frac{1}{\sqrt{q!}} \begin{pmatrix} v^{(1')} v^{(2')} \dots v^{(q')} \\ y_{n+1} y_{n+2} \dots y_{N_0} \end{pmatrix};$$

$$(v^{(k')} y_h) = v_{\alpha_h'}^{(k')}(y_h) \,.$$

(The adjoints $\bar{\psi}$ are defined, for reasons of relativistic convenience, by writing in the Slater determinants the adjoints $\bar{u}$, $\bar{v}$ of the single-particle wave functions u, v).

§ 2·4. – We can, finally, collect our results in the desired form. Define a kernel as the function:

$$(30)\qquad K^{\mu_1', \mu_2', \dots, \mu_{N_0}'}_{\beta_1', \beta_2', \dots, \beta_{N_0}'; \alpha_1', \alpha_2', \dots, \alpha_{N_0}'} \begin{pmatrix} x_1 x_2 \dots x_{N_0} \\ y_1 y_2 \dots y_{N_0} \end{pmatrix} t_1 t_2 \dots t_{P_0}) =$$

$$= \sum_{N(P_0)} \frac{\lambda^N}{N!} \int \sum \gamma^{\mu_1}_{\alpha_1\beta_1} \dots \gamma^{\mu_N}_{\alpha_N\beta_N} \begin{pmatrix} x_1 x_2 \dots x_{N_0} \xi_1 \xi_2 \dots \xi_N \\ y_1 y_2 \dots y_{N_0} \xi_1 \xi_2 \dots \xi_N \end{pmatrix} \; t_1 t_2 \dots t_{P_0} \xi_1 \xi_2 \dots \xi_N] ;$$

in (30), as in (2), $\int \sum$ means integration and summation over all the variables $\xi_1, \xi_2, ..., \xi_N$ and related (unprimed) indices; thus, K is a (tensor) function of $x_1, ..., t_{P_0}$ and related (primed) indices. Call Φ_I and Φ_F the Fock-space wave functions corresponding to the initial and final states, inclusive of all kinds of particles, normalized to unity: that is, Φ_I is the product of (19), (26) and (27), Φ_F of (20), (28) and (29). Then (2) can be written as

$$M_{FI}(t_1, t_0) = C_{FI} \int^{\wedge} \bar{\Phi}_F K \begin{pmatrix} x_1 x_2 \dots x_{N_0} \\ y_1 y_2 \dots y_{N_0} \end{pmatrix} \Big| \, t_1 t_2 \dots t_{P_0} \Big) \Phi_I \,, \tag{31}$$

with:

$$C_{FI} = (-1)^{P_{FI}} \frac{2^{P_0/2} \sqrt{[E]_b \cdot [E]_f}}{m^{2N_0} \cdot \sqrt{a!\, b!} \cdot \sqrt{m!\, n!\, p!\, q!}} \,. \tag{32}$$

Formula (31) solves completely the problem proposed in this Part; it also shows, on inspection, the relativistic covariance of the formalism in a direct way. Henceforth we shall study exclusively the kernels K, whose properties will be studied in detail in Part **4** (keeping in mind, however, what has been stated at the beginning).

3. – Two Particular Cases.

It is of interest, at this point, to digress briefly from our main topic, in order to show how this formalism works in simpler and well-known instances. This, besides providing a check on the numerical part of our work, may serve to give some insight into the mathematical nature of the quantities we are considering. We treat in the next sections the particular cases in which only the boson, or only the fermion field is quantized; the well-known results of FEYNMAN ([3]), GLAUBER ([4]), SALAM and MATTHEWS ([6]), and others, will be seen to follow in a direct and simple way.

3·1. *Boson field quantized.* – As is to be expected from physics (the interaction representation vacuum is the real vacuum, since no vacuum polarization is possible) this case is the simpler one: the perturbation expansion can be summed up exactly. The situation of the particle field is described by a classical current $j_\mu(x)$, so that the interaction Hamiltonian is

$$H_{\text{int}} = - j_\mu(x) A_\mu(x)$$

([6]) A. SALAM and P. T. MATTHEWS: *Phys. Rev.*, **90**, 690 (1953).

and the U-matrix element between the initial and final photon states is immediately seen to be:

$$M_{FI} = \sum_{N=0}^{\infty} \frac{i^N}{N!} \int \Sigma \left\langle \Psi_F^{(b)} \; T\left(\prod_{i=1}^{N} j_{\mu_i}(\xi_i) A_{\mu_i}(\xi_i) \right) \Psi_I^{(b)} \right\rangle =$$

$$= \frac{2^{P_0/2} \sqrt{[E]_b}}{\sqrt{a!\, b!}} \int \hat{\varphi}_F^{(b)*} \bar{K}(t_1 t_2 \dots t_{P_0}) \varphi_I^{(b)} \, ,$$

from (2), (19), (20); we have set:

$$(33) \qquad \bar{K}(t_1 t_2 \dots t_{P_0}) = \sum_{N(P_0)} \frac{i^N}{N!} \int \Sigma \, j_{\mu_1}(\xi_1) \dots j_{\mu_N}(\xi_N) [t_1 t_2 \dots t_{P_0} \xi_1 \xi_2 \dots \xi_N] \, .$$

Consider first M_{00}:

$$(34) \qquad M_{00} = \bar{K}_0 = \sum_{n=0}^{\infty} \frac{i^{2n}}{(2n)!} \int \Sigma \, j_{\mu_1}(\xi_1) \dots j_{\mu_{2n}}(\xi_{2n}) [\xi_1 \xi_2 \dots \xi_{2n}] =$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^n n!} \left[\int \mathrm{d}\xi_1 \int \mathrm{d}\xi_2 \sum_{\mu_1 \mu_2} j_{\mu_1}(\xi_1) j_{\mu_2}(\xi_2) [\xi_1 \xi_2] \right]^n =$$

$$= \exp\left[-\frac{1}{4} \int \mathrm{d}\xi_1 \int \mathrm{d}\xi_2 \sum_{\mu} j_\mu(\xi_1) j_\mu(\xi_2) D^F(\xi_1 - \xi_2) \right] ,$$

which is Glauber's result, on extending the integrations to all space-time. In the evaluation of (34), use has been made of the symmetry properties of hafnians established in I (see, in particular, formula (I.36)).

Consider now, quite in general, the kernel defined by (33). On expanding the hafnians therein contained by the elements of their first lines one finds, again by using their symmetry properties, that:

$$(35) \qquad \bar{K}(t_1 t_2 \dots t_{P_0}) =$$

$$= \sum_{h=2}^{P_0} [t_1 t_h] \cdot \bar{K}(t_2 \dots t_{h-1} t_{h+1} \dots t_{P_0}) + i \int \sum_{\mu_1} \mathrm{d}\xi_1 [t_1 \xi_1] \, j_{\mu_1}(\xi_1) \cdot \bar{K}(t_2 t_3 \dots t_{P_0}) \, .$$

((35) is a first example of the type of relations deduced in Part **4** for the general case: see, in particular, formula (48). A detailed argument is given there). We propose to study only the case treated by GLAUBER, i.e. $a=0$, $b=P_0$: A non-vanishing contribution to the matrix element can come only from the second term on the right side of (35); so that, by iteration, the relevant part

of the kernel reduces to:

$$i^{P_0}\bar{K}_0 \cdot \int \sum_{\mu_1} \mathrm{d}\xi_1 [t_1 \xi_1] j_{\mu_1}(\xi_1) \cdot \ldots \cdot \int \sum_{\mu_{P_0}} \mathrm{d}\xi_{P_0} [t_{P_0} \xi_{P_0}] j_{\mu_{P_0}}(\xi_{P_0}) \,.$$

and

$$M_{P_0,0} = \frac{2^{P_0/2}\sqrt{[E]_b}}{\sqrt{P_0!}} \int^{\wedge} \varphi_{P_0}^{(b)*} \bar{K}(t_1 t_2 \ldots t_{P_0}) =$$

$$= \frac{i^{P_0} 2^{P_0/2} \sqrt{[E]_b}}{\sqrt{\sigma_{1'}! \, \sigma_{2'}! \ldots \sigma_{\beta'}! \, P_0!}} \bar{K}_0 \int^{\wedge} \left\{ \begin{bmatrix} \zeta^{(1')*} & \zeta^{(2')*} & \ldots & \zeta^{(P_0')*} \\ t_1 & t_2 & \ldots & t_{P_0} \end{bmatrix} \cdot \right.$$

$$\left. \cdot \int \sum_{\mu_1} \mathrm{d}\xi_1 [t_1 \xi_1] j_{\mu_1}(\xi_1) \cdot \ldots \cdot \int \sum_{\mu_{P_0}} \mathrm{d}\xi_{P_0} [t_{P_0} \xi_{P_0}] j_{\mu_{P_0}}(\xi_{P_0}) \right\} =$$

$$= \frac{i^{P_0} \cdot \bar{K}_0}{\sqrt{\sigma_{1'}! \, \sigma_{2'}! \ldots \sigma_{\beta'}!}} \left\{ \int \sum_{\mu_1'} \mathrm{d}\xi_1 z_{\mu_1'}^{(1')*}(\xi_1) j_{\mu_1'}(\xi_1) \right\}^{\sigma_{1'}} \ldots \left\{ \int \sum_{\mu_{\beta'}'} \mathrm{d}\xi_{\beta'} z_{\mu_{\beta'}'}^{(\beta')*}(\xi_{\beta'}) j_{\mu_{\beta'}'}(\xi_{\beta'}) \right\}^{\sigma_{\beta'}} . \tag{36}$$

We have used here (15), (16), (20) and the symmetry properties of permanents defined in I. We could have done all this more simply, without passing through kernels and with a reduced notation; our purpose, however, has been to check the numerical work of Part 2. (36) is the U-matrix element for the creation of P_0 photons, of which $\sigma_{1'}$ in state $1'$ with wave function $z^{(1')}$, etc. If we wish to calculate the probability that P_0 photons are emitted, we must sum $|M_{P_0,0}|^2$ over all possible photon states. If we divide each $|M_{P_0,0}|^2$ by the statistical weight $P_0!/\sigma_{1'}! \, \sigma_{2'}! \ldots \sigma_{\beta'}!$ of the corresponding final states, we can sum, into each of the integrals

$$\iint \sum_{\mu_1 \mu_2}' \mathrm{d}\xi_1 \mathrm{d}\xi_2 z_{\mu_1'}^{(\lambda')}(\xi_1) z_{\mu_2'}^{(\lambda')*}(\xi_2) j_{\mu_1'}(\xi_1) j_{\mu_2'}(\xi_2) \,,$$

which appear in $|M_{P_0,0}|^2$, over all the states λ'; the result is then, again with the integration extended to all space-time:

$$w_{P_0} = \sum |M_{P_0,0}|^2 = \frac{1}{P_0!} |\bar{K}_0|^2 \cdot \left\{ \frac{1}{2} \iint \mathrm{d}\xi_1 \mathrm{d}\xi_2 \, D^F(\xi_1 - \xi_2) \sum_{\mu} j_\mu(\xi_1) j_\mu(\xi_2) \right\}^{P_0} , \tag{37}$$

which is, again, Glauber's result. One may check, with the same technique, the unitarity condition: for instance, it is easily seen that:

$$\langle 0 | U^+ U | 0 \rangle = \sum_{P_0} w_{P_0} = 1 \,. \tag{38}$$

Also the results obtained by Glauber for more general interactions could be derived in the same manner; this lies, however, beyond our present scope.

3·2. *Fermion field quantized.* – We treat this case only in a most brief manner, since all the relevant consequences which we might wish to study have been already deduced, essentially in the same way, by SALAM and MATTHEWS (6): we refer the reader to their excellent work for more details. It will suffice to consider only processes involving two real fermions. One finds:

$$(39) \qquad M_{FI} = \sum_{N=0}^{\infty} \frac{\lambda^N}{N!} \int \sum \boldsymbol{A}(\xi_1) \dots \boldsymbol{A}(\xi_N) \begin{pmatrix} \xi_1 \xi_2 \dots \xi_N v \\ \xi_1 \xi_2 \dots \xi_N u \end{pmatrix} = \frac{E_1 E_2}{m^2} \int v(x) \widetilde{K} \begin{pmatrix} x \\ y \end{pmatrix} u(y) \, ,$$

where, for short: $\boldsymbol{A}(\xi) = \sum_{\mu} A_\mu(x) \gamma^\mu_{\alpha\beta}$ and all spinor indices are dropped; $\sum_N$ ranges over *all* the values $N = 0, 1, 2, 3, \dots$, since the boson field is not quantized; $\widetilde{K}$ is defined as

$$(40) \qquad \widetilde{K} \begin{pmatrix} x \\ y \end{pmatrix} = \sum_{N=0}^{\infty} \frac{\lambda^N}{N!} \int \sum \boldsymbol{A}(\xi_1) \dots \boldsymbol{A}(\xi_N) \begin{pmatrix} \xi_1 \xi_2 \dots \xi_N x \\ \xi_1 \xi_2 \dots \xi_N y \end{pmatrix} ,$$

from which (compare with (49)):

$$(41) \qquad \widetilde{K} \begin{pmatrix} x \\ y \end{pmatrix} = \widetilde{K}_0 \cdot \frac{1}{2} S^F(x - y) - \frac{\lambda}{2} \int \mathrm{d}\xi S^F(x - \xi) \boldsymbol{A}(\xi) \widetilde{K} \begin{pmatrix} \xi \\ y \end{pmatrix} ,$$

where:

$$(42)^1 \qquad \widetilde{K}_0 = \sum_{N=0}^{\infty} \frac{\lambda^N}{N!} \int \sum \boldsymbol{A}(\xi_1) \dots \boldsymbol{A}(\xi_N) \begin{pmatrix} \xi_1 \xi_2 \dots \xi_N \\ \xi_1 \xi_2 \dots \xi_N \end{pmatrix} = e^{-L} ,$$

the last being the expression given by FEYNMAN (3) for the vacuum-vacuum transition amplitude (as follows also immediately from our general formalism); its typical exponential form is a simple consequence of the Fredholm theory.

4. – The Branching Equatoins.

4·1. – The starting point in the deduction of the branching equations is the formal expansion (30) of the kernels K. Since the structure of these depends only upon the two numbers N_0 and P_0, we denote them, when their explicit form is not needed, simply with K_{N_0,P_0}; we also drop the indices which specify their spinor and vector components, with the understanding that to each variable x_h, y_h, t_h is associated, respectively, a non written index β'_h, α'_h, μ'_h. K_{N_0,P_0} describes, to all orders, processes in which $2N_0$ fermions and P_0 bosons occur as real particles; it may be convenient to associate to it a « total graph » showing P_0 dotted undirected lines (*t-lines*), N_0 solid lines directed outwardly (*x-lines*) and N_0 solid lines directed inwardly (*y-lines*) of some closed plane region, with the Feynman convention that a created (destroyed) electron

is represented by an $x(y)$-line, a created (destroyed) positron is represented by an $y(x)$-line and a created or destroyed photon by a t-line; it being irrelevant in which points these lines meet the boundary of the region. With these graphs the branching equations acquire an immediate intuitive meaning, as will be shown in some cases.

Fundamental for the following are the symmetry properties of the kernels with respect to permutations of their variables; these refer to the exchange of variables x_h, y_h, or t_h *and* of the corresponding indices β'_h, α'_h, μ'_h; same for dummy variables and indices under integral sign. Concisely: one has to study the behavior of K under permutations of the *labels* which characterize both a variable x (or y, or t, or ξ) and the corresponding index. From (30), and from the properties of determinants and hafnians, it is then easily seen that:

(43) *K_{N_0,P_0} is antisymmetric under permutations of the labels affixed to the variables $x(y)$ and to the corresponding indices $\beta'(\alpha')$;*

(44) *K_{N_0,P_0} is symmetric under permutations of the labels affixed to the variables t and to the corresponding indices μ'.*

For each dummy variable ξ_h, there are three dummy indices, α_h, β_h, μ_h, with the same label h: clearly, labels can be permuted at will, here, without changing the value of K_{N_0,P_0}; more than this:

(45) *Each integrand in the expansion* (30) *remains unchanged under permutations of the labels affixed to the dummy variables ξ and to the corresponding indices α, β, μ.*

These obvious properties have been already used by us several times (for instance, in the derivation of formula (I.36)); their explicit mention is, however, necessary for our purposes. They are expected to play a relevant role, together with the properties of Lorentz covariance, also in the actual solution of the equations obtained in this work. We end this section by listing two more important properties which hold for total graphs *only if there are no external fields*; for the proof, see Appendix III. The first is the well-known *Furry theorem*:

$$K(t_1 t_2 \dots t_{2h+1}) \equiv 0 \,; \tag{46}$$

the second, a generalization of the first, is a *saturation property*:

$$\sum_{(\text{all } \alpha',\, \beta')} \gamma^{\mu''_1}_{\alpha'_1\beta'_1} \gamma^{\mu''_2}_{\alpha'_2\beta'_2} \dots \gamma^{\mu''_{N_0}}_{\alpha'_{N_0}\beta'_{N_0}} K\begin{pmatrix} x_1 x_2 \dots x_{N_0} \\ x_1 x_2 \dots x_{N_0} \end{pmatrix} t_1 t_2 \dots t_{P_0} \equiv 0 \,, \tag{47}$$

whenever $N_0 + P_0 = odd$, regardless of the values of the indices $\mu''_1, \dots, \mu'_{N_0}$; also, therefore, if these are saturated with the corresponding indices of K,

which is the case of interest. If one starts from the branching equations listed below, (46) is a consequence of (47).

4·2. – The technique adopted in this and in the next section is an extension of the one familiar in the formal study of the Fredholm theory, the difference being that we have, besides determinants, also hafnians to deal with; it will lead to systems of infinite equations connecting kernels with different N_0 and P_0. We derive in this section those of them which involve only K's, in the next those containing also derivatives of the K's with respect to λ.

Let us examine first what can be had from the symmetry properties of hafnians. From

$$[t_1 t_2 \dots t_{P_0} \xi_1 \xi_2 \dots \xi_N] = \sum_{h=2}^{P_0} [t_1 t_h][t_2 \dots t_{h-1} t_{h+1} \dots t_{P_0} \xi_1 \xi_2 \dots \xi_N] + \\ + \sum_{h=1}^{N} [t_1 \xi_h][t_2 \dots t_{P_0} \xi_1 \dots \xi_{h-1} \xi_{h+1} \dots \xi_N]$$

one finds, by applying (45):

$$(48) \qquad K\left(\begin{matrix} x_1 x_2 \dots x_{N_0} \\ y_1 y_2 \dots y_{N_0} \end{matrix} \middle| t_1 t_2 \dots t_{P_0}\right) = \sum_{h=2}^{P_0} [t_1 t_h] K\left(\begin{matrix} x_1 x_2 \dots x_{N_0} \\ y_1 y_2 \dots y_{N_0} \end{matrix} \middle| t_2 \dots t_{h-1} t_{h+1} \dots t_{P_0}\right) + \\ + \lambda \int \sum_1 \mathrm{d}\xi_1 \gamma^1 [t_1 \xi_1] K\left(\begin{matrix} x_1 x_2 \dots x_{N_0} \xi_1 \\ y_1 y_2 \dots y_{N_0} \xi_1 \end{matrix} \middle| t_2 \dots t_{P_0}\right)$$

where we write, for short, γ^1 and $\sum_1$ for $\gamma^{\mu_1}_{\alpha_1 \beta_1}$ and $\sum_{\alpha_1, \beta_1, \mu_1}$; the dummy indices α_1, β_1, μ_1 are those corresponding to the variable ξ_1 (likewise in the following). (48), as well as the other equations deduced below, does not show explicitly the symmetry properties stated in (43) or (44); these do, however, reappear after a convenient number of iterations. This fact does not interest us here; rather, *we consider* (43) *and* (44) *as conditions which, together with the branching equations, define the kernels.* We are thus, clearly, spared the trouble of re-writing, say, (48) with each of the indices t_2, t_3, ..., t_{P_0} put explicitly into evidence. Fig. 1 shows the interpretation of (48) in terms of total graphs. Formula (48) is remarkable, since it shows that K_{N_0, P_0} can be obtained by means of simple integrations if $K_{N_0, P_0 - 2}$ and $K_{N_0 + 1, P_0 - 1}$ are known; so that, by a finite process of iteration, one can determine K_{N_0, P_0} if the « purely fermionic kernels » (with $P_0 = 0$) $K_{N_0'}$ are known for all the values of N_0' ranging from N_0 to $N_0 + P_0$. (Questions of convergence are beyond the scope of the present discussion). It suffices therefore to de-

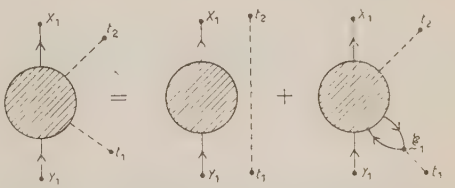

Fig. 1. – Total-graph interpretation of eq. (48), for: $N_0 = 1$, $P_0 = 2$ space-time integration over ξ_1 is required.

termine the purely fermionic kernels $K\begin{pmatrix} x_1 x_2 \dots x_{N_0} \\ y_1 y_2 \dots y_{N_0} \end{pmatrix}$ to get the general solution of the problem; this fact reduces greatly the difficulties one is faced with, as it reduces a double infinity of unknown functions to a single infinity. We shall therefore mostly concentrate on equations among K_{N_0}; it is convenient, however, to have first the complete sets of possible branching equations.

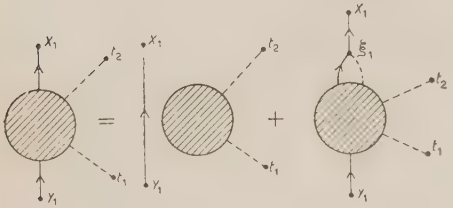

Fig. 2. – Same case of Fig. 1, for eq. (49).

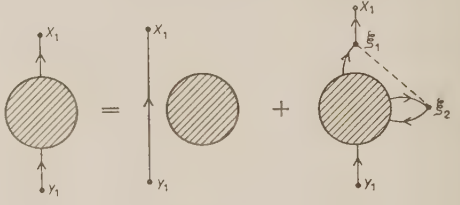

Fig. 3. – Total-graph interpretation of eq. (51), for $N_0=1$ ($P_0=0$): integration over ξ_1 and ξ_2.

Let us now operate in a similar way upon the determinants. From:

$$\begin{pmatrix} x_1 x_2 \dots x_{N_0} \xi_1 \xi_2 \dots \xi_N \\ y_1 y_2 \dots y_{N_0} \xi_1 \xi_2 \dots \xi_N \end{pmatrix} = \sum_{h=1}^{N_0} (-1)^{h-1} (x_1 y_h) \begin{pmatrix} x_2 \dots x_h \quad x_{h+1} \dots x_{N_0} \xi_1 \xi_2 \dots \xi_N \\ y_1 \dots y_{h-1}\, y_{h+1} \dots y_{N_0} \xi_1 \xi_2 \dots \xi_N \end{pmatrix} -$$

$$- \sum_{h=1}^{N} (x_1 \xi_h) \begin{pmatrix} \xi_h x_2 \dots x_{N_0}\ \xi_1 \dots \xi_{h-1}\ \xi_{h+1} \dots \xi_N \\ y_1 y_2 \dots y_{N_0}\ \xi_1 \dots \xi_{h-1}\ \xi_{h+1} \dots \xi_N \end{pmatrix},$$

we get, with the same procedure (see Fig. 2):

$$(49) \quad K\left(\begin{matrix} x_1 x_2 \dots x_{N_0} \\ y_1 y_2 \dots y_{N_0} \end{matrix} \middle| \, t_1 t_2 \dots t_{P_0}\right) = \sum_{h=1}^{N_0} (-1)^{h-1} (x_1 y_h) K\left(\begin{matrix} x_2 \dots x_h \quad x_{h+1} \dots x_{N_0} \\ y_1 \dots y_{h-1} y_{h+1} \dots y_{N_0} \end{matrix} \middle| \, t_1 t_2 \dots t_{P_0}\right) -$$

$$- \lambda \int \sum_1 \mathrm{d}\xi_1 \gamma^1 (x_1 \xi_1) K\left(\begin{matrix} \xi_1 x_2 \dots x_{N_0} \\ y_1 y_2 \dots y_{N_0} \end{matrix} \middle| \, \xi_1 t_1 t_2 \dots t_{P_0}\right)$$

and, on expanding by the elements of the first columns of the determinants instead of the first rows, the associate equations:

$$(50) \quad K\left(\begin{matrix} x_1 x_2 \dots x_{N_0} \\ y_1 y_2 \dots y_{N_0} \end{matrix} \middle| \, t_1 t_2 \dots t_{P_0}\right) = \sum_{h=1}^{N_0} (-1)^{h-1} (x_h y_1) K\left(\begin{matrix} x_1 \dots x_{h-1} x_{h+1} \dots x_{N_0} \\ y_2 \dots y_h \quad y_{h+1} \dots y_{N_0} \end{matrix} \middle| \, t_1 t_2 \dots t_{P_0}\right) -$$

$$- \lambda \int \sum_1 \mathrm{d}\xi_1 \gamma^1 (\xi_1 y_1) K\left(\begin{matrix} x_1 x_2 \dots x_{N_0} \\ \xi_1 y_2 \dots y_{N_0} \end{matrix} \middle| \, \xi_1 t_1 t_2 \dots t_{P_0}\right).$$

By combining (48) with (49) and (50) we obtain, finally, the equations for purely fermionic kernels, which connect K_{N_0} with K_{N_0-1} and K_{N_0+1}

(see Fig. 3):

$$(51)\qquad K\begin{pmatrix} x_1 x_2 \dots x_{N_0} \\ y_1 y_2 \dots y_{N_0} \end{pmatrix} = \sum_{h=1}^{N_0} (-1)^{h-1}(x_1 y_h) K\begin{pmatrix} x_2 \dots x_h \quad x_{h+1} \dots x_{N_0} \\ y_1 \dots y_{h-1}\, y_{h+1} \dots y_{N_0} \end{pmatrix} - \\ - \lambda^2 \iint \sum_{1,2} \mathrm{d}\xi_1 \mathrm{d}\xi_2 \gamma^1 \gamma^2 (x_1 \xi_1)[\xi_1 \xi_2] K\begin{pmatrix} \xi_1 x_2 \dots x_{N_0} \xi_2 \\ y_1 y_2 \dots y_{N_0} \xi_2 \end{pmatrix},$$

$$(52)\qquad K\begin{pmatrix} x_1 x_2 \dots x_{N_0} \\ y_1 y_2 \dots y_{N_0} \end{pmatrix} = \sum_{h=1}^{N_0} (-1)^{h-1}(x_h y_1) K\begin{pmatrix} x_1 \dots x_{h-1}\, x_{h+1} \dots x_{N_0} \\ y_2 \dots y_h \quad y_{h+1} \dots y_{N_0} \end{pmatrix} - \\ - \lambda^2 \iint \sum_{1,2} \mathrm{d}\xi_1 \mathrm{d}\xi_2 \gamma^1 \gamma^2 (\xi_1 y_1)[\xi_1 \xi_2] K\begin{pmatrix} x_1 x_2 \dots x_{N_0} \xi_2 \\ \xi_1 y_2 \dots y_{N_0} \xi_2 \end{pmatrix}.$$

It follows from (51) or (52) that, for $\lambda = 0$:

$$(53)\qquad \underset{(\lambda=0)}{K}\begin{pmatrix} x_1 x_2 \dots x_{N_0} \\ y_1 y_2 \dots y_{N_0} \end{pmatrix} = \begin{pmatrix} x_1 x_2 \dots x_{N_0} \\ y_1 y_2 \dots y_{N_0} \end{pmatrix},$$

as was expected.

4·3. – The equations containing λ-derivatives are obtained, in a quite similar way, by expanding hafnians and determinants by the line ξ_1: From:

$$[t_1 t_2 \dots t_{P_0}\, \xi_1 \xi_2 \dots \xi_N] = \sum_{h=2}^{N} [\xi_1 \xi_h][t_1 \dots t_{P_0}\, \xi_2 \dots \xi_{h-1}\, \xi_{h+1} \dots \xi_N] + \\ + \sum_{h=1}^{P_0} [t_h \xi_1][t_1 \dots t_{h-1}\, t_{h+1} \dots t_{P_0}\, \xi_2 \dots \xi_{P_0}]$$

one finds:

$$(54)\qquad \frac{\mathrm{d}}{\mathrm{d}\lambda} K\left(\begin{matrix} x_1 x_2 \dots x_{N_0} \\ y_1 y_2 \dots y_{N_0} \end{matrix}\middle| t_1 t_2 \dots t_{P_0}\right) = \\ = \sum_{h=1}^{P_0} \int \sum_{1} \mathrm{d}\xi_1 \gamma^1 [t_h \xi_1] K\left(\begin{matrix} x_1 x_2 \dots x_{N_0} \xi_1 \\ y_1 y_2 \dots y_{N_0} \xi_1 \end{matrix}\middle| t_1 \dots t_{h-1}\, t_{h+1} \dots t_{P_0}\right) + \\ + \lambda \iint \sum_{1,2} \mathrm{d}\xi_1 \mathrm{d}\xi_2 \gamma^1 \gamma^2 [\xi_1 \xi_2] K\left(\begin{matrix} x_1 x_2 \dots x_{N_0} \xi_1 \xi_2 \\ y_1 y_2 \dots y_{N_0} \xi_1 \xi_2 \end{matrix}\middle| t_1 t_2 \dots t_{P_0}\right)$$

and, in particular:

$$(55)\qquad \frac{\mathrm{d}}{\mathrm{d}(\lambda^2/2)} K\begin{pmatrix} x_1 x_2 \dots x_{N_0} \\ y_1 y_2 \dots y_{N_0} \end{pmatrix} = \iint \sum_{1,2} \mathrm{d}\xi_1 \mathrm{d}\xi_2 \gamma^1 \gamma^2 [\xi_1 \xi_2] K\begin{pmatrix} x_1 x_2 \dots x_{N_0} \xi_1 \xi_2 \\ y_1 y_2 \dots y_{N_0} \xi_1 \xi_2 \end{pmatrix}.$$

When expanding determinants, one can choose either the row ξ_1 or the

column ξ_1; the result is given, respectively, by the associate equations:

$$(56)\qquad \frac{\mathrm{d}}{\mathrm{d}\lambda} K\begin{pmatrix} x_1 x_2 \dots x_{N_0} \\ y_1 y_2 \dots y_{N_0} \end{pmatrix} t_1 t_2 \dots t_{P_0}\Big) = \int \sum_1 \mathrm{d}\xi_1 \gamma^1(\xi_1\xi_1) K\begin{pmatrix} x_1 x_2 \dots x_{N_0} \\ y_1 y_2 \dots y_{N_0} \end{pmatrix} \xi_1 t_1 t_2 \dots t_{P_0}\Big) -$$
$$- \sum_{h=1}^{N_0} \int \sum_1 \mathrm{d}\xi_1 \gamma^1(\xi_1 y_h) K\begin{pmatrix} x_1 \dots x_{h-1}\, x_h x_{h+1} \dots x_{N_0} \\ y_1 \dots y_{h-1}\, \xi_1 y_{h+1} \dots y_{N_0} \end{pmatrix} \xi_1 t_1 \dots t_{P_0}\Big) -$$
$$- \lambda \iint \sum_{1,2} \mathrm{d}\xi_1 \mathrm{d}\xi_2 \gamma^1\gamma^2(\xi_1\xi_2) K\begin{pmatrix} x_1 x_2 \dots x_{N_0}\xi_2 \\ y_1 y_2 \dots y_{N_0}\xi_1 \end{pmatrix} \xi_1\xi_2 t_1 t_2 \dots t_{P_0}\Big),$$

$$(57)\qquad \frac{\mathrm{d}}{\mathrm{d}\lambda} K\begin{pmatrix} x_1 x_2 \dots x_{N_0} \\ y_1 y_2 \dots y_{N_0} \end{pmatrix} t_1 t_2 \dots t_{P_0}\Big) = \int \sum_1 \mathrm{d}\xi_1 \gamma^1(\xi_1\xi_1) K\begin{pmatrix} x_1 x_2 \dots x_{N_0} \\ y_1 y_2 \dots y_{N_0} \end{pmatrix} \xi_1 t_1 t_2 \dots t_{P_0}\Big) -$$
$$- \sum_{h=1}^{N_0} \int \sum_1 \mathrm{d}\xi_1 \gamma^1(x_h\xi_1) K\begin{pmatrix} x_1 \dots x_{h-1}\, \xi_1 x_{h+1} \dots x_{N_0} \\ y_1 \dots y_{h-1}\, y_h y_{h+1} \dots y_{N_0} \end{pmatrix} \xi_1 t_1 \dots t_{P_0}\Big) -$$
$$- \lambda \iint \sum_{1,2} \mathrm{d}\xi_1 \mathrm{d}\xi_2 \gamma^1\gamma^2(\xi_1\xi_2) K\begin{pmatrix} x_1 x_2 \dots x_{N_0}\xi_2 \\ y_1 y_2 \dots y_{N_0}\xi_1 \end{pmatrix} \xi_1\xi_2 t_1 \dots t_{P_0}\Big).$$

The first term on the right side of (56) and (57) vanishes if there are no external fields, as a consequence of the Heisenberg prescription (see I).

4·4. – The equations listed in sections **4·2** and **4·3**, together with the conditions given in section **4·1**, constitute a *complete set*: any other equation, of whatsoever complication, which can be derived from the perturbative expansions is, necessarily, a consequence of the equations and conditions of this set (7). One might, at first, argue that if hafnians and determinants are expanded in some other way, for instance by minors of any order (or, the second, by elements of rows *and* columns, or, worse, if both hafnians and determinants are expanded simultaneously), much more complicated equations would be obtained, including λ-derivatives to any wished order. Clearly, however, any such expansion can be reproduced, step-wise, by means of expansions by single lines, iterated a sufficient number of times: that is, all such « more complicated equations » are necessarily obtainable from those of our set by iteration. Or one may attempt to obtain relations from (30) by some other procedure; it is an immediate consequence of (30), for instance, that, denoting with K_0 the kernel for $N_0 = P_0 = 0$:

$$\int \sum_1 \mathrm{d}\xi_1 \gamma^1 K\begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} \xi_1\Big) = \frac{\mathrm{d}K_0}{\mathrm{d}\lambda};$$

(7) Relations among, and conditions on kernels deriving from invariance requirements (in particular, gauge invariance) are *not* considered in the present general discussion, because they are specific to each theory: we are concerned here mostly with properties common to *all* two-field theories of the type currently envisaged.

this equation does not appear among those written before. But, from (48)

$$K\begin{pmatrix}\xi_1 & \xi_1\\ \xi_1\end{pmatrix} = \lambda\int \sum_2 \mathrm{d}\xi_2\gamma^2[\xi_1\xi_2]K\begin{pmatrix}\xi_1\xi_2\\ \xi_1\xi_2\end{pmatrix},$$

so that, actually, the equation in question is a particular case of (55), in virtue of (48). The analogy with the case of the Fredholm theory suffices to answer this doubt in general.

It appears, therefore, that our statement regarding the completeness of the set is justified. This does not mean, of course, that, by means of procedures other than the one followed here—such as taking, for instance, the equations which define U as the starting point, rather than considering their formal perturbative solutions—one might not get different sets of equations, for kernels defined in a different manner. Ultimately, however, any attempt of the sort would amount simply to a re-formulation of the problem, and should, necessarily, prove equivalent to the one followed here. It follows that, provided we are assured: *a*) that the solutions of our equations reduce to the perturbative expansions (30) formally (and significantly if these converge), and *b*) that our equations are correct independently of the existence of perturbative solutions, they can be taken as the adequate starting point for a quantitative, more than formal investigation.

a) That the branching equations admit of solutions of the form (30) is self-evident and need not be proved, since we have started from (30) to deduce them. Uniqueness can be tested by direct evaluation; the check is particularly simple in the case of eq. (55), with the initial conditions (53) : (30) can be written (see (I.36)):

$$(58)\qquad K\begin{pmatrix}x_1x_2\dots x_{N_0}\\ y_1y_2\dots y_{N_0}\end{pmatrix} = \sum_{n=1}^{\infty}\frac{1}{n!}\begin{pmatrix}\lambda^2\\ 2\end{pmatrix}^n\int\sum\gamma^1\dots\gamma^{2n}[\xi_1\xi_2]\dots[\xi_{2n-1}\xi_{2n}]\cdot\begin{pmatrix}x_1x_2\dots x_{N_0}\xi_1\xi_2\dots\xi_{2n}\\ y_1y_2\dots y_{N_0}\xi_1\xi_2\dots\xi_{2n}\end{pmatrix};$$

to get (30), one has only to evaluate the McLaurin expansion of K_{N_0}, using (55) and (53).

K_0 is defined by the condition $K_0(\lambda=0)=1$ and by (55). This is the only case in which we need consider the equations with λ-derivatives. The equations of section 4·2 do, otherwise, fully suffice; they are the less restrictive, since the λ-differentiability of the kernels is not required for their validity. It is interesting that they give directly the ratios $K_{N_0,P_0}/K_0$, the only quantities of interest if there are no external fields; their consideration can be therefore quite useful in the study of the convergence and renormalization

of the perturbative expansions (30), because this fact permits to disentangle the factor K_0 out of the series (30)—a fatiguing algebraic problem if attacked directly. We shall return briefly on this question in Part 5.

b) That the branching equations, or consequences of them, may be deduced directly from the foundations of the theory, without the intermediary of the perturbative expansions, can be seen in several ways. This question is discussed here only very briefly, on a simple example, since we hold the thing to be evident on physical grounds: the interpretation of the equations in terms of total graphs exhibits, indeed, the expected behaviour of a two-field theory. Consider K_0, the vacuum-vacuum transition amplitude; its perturbative expansion is given by (I.35), or (58) with $N_0=0$. We shall first refer to this expansion and see how the kernels can be deduced from it alone by means of suitable variational differentiations; it will be then evident that the procedure does not require the explicit knowledge of K_0 and, conversely, yields equations of the branching type for kernels defined independently of perturbation theory. Our treatment is, essentially, a simplified version of more general considerations due to FEYNMAN (8). It will suffice to consider the case of $K(t_1t_2)$. Using (58) and (9), we find easily that:

$$\frac{\delta^2 K_0}{\delta z^{(\varrho)}_{\mu_1}(\tau_1)\,\delta z^{(\varrho)*}_{\mu_2}(\tau_2)} = \frac{\delta K_0}{\delta[\tau_1\tau_2]} = \frac{\lambda^2}{2}\sum_{1,2}\gamma^1\gamma^2 K\begin{pmatrix}\tau_1\tau_2\\ \tau_1\tau_2\end{pmatrix}; \tag{59}$$

this expression is independent of ϱ; in deriving it, we have differentiated each term of the expansion of K_0, after substituting the functions $[\xi'\xi'']$ with their expansions (9) (this gives a factor N to each term). The interpretation of (59) is quite simple: it represents the total vacuum-vacuum graph, out of which the photon line connecting the points τ_1 and τ_2 has been removed away. To build $K(t_1t_2)$ from (59), thinking in terms of total graphs, we have, clearly, to attach in t_1 and t_2 the propagation functions $[t_1\tau_1]$ and $[t_2\tau_2]$, or $[t_1\tau_2]$ and $[t_2\tau_1]$ (we must therefore sum over these two possibilities) and to integrate over all possible τ_1 and τ_2; finally, we have to add a term representing the undisturbed propagation of a real photon. The result is

$$K(t_1t_2) = K_0\cdot[t_1t_2] + 2\iint [t_1\tau_1][t_2\tau_2]\frac{\delta K_0}{\delta[\tau_1\tau_2]}\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 = \tag{60}$$

$$= K_0[t_1t_2] + \lambda^2\iint\sum_{1,2}\gamma^1\gamma^2[t_1\xi_1][t_2\xi_2]K\begin{pmatrix}\xi_1\xi_2\\ \xi_1\xi_2\end{pmatrix},$$

(8) R. P. FEYNMAN: *Phys. Rev.*, **80**, 440 (1950). See also: J. L. ANDERSON: Rep. ONR 594 (00), Un. of Maryland.

which is just what (48) would give, after iteration. We may now forget about the perturbation expansion of K_0 and read (59) from right to left, thinking that K_0 is some functional of the functions $z^{(\varrho)}$ and $z^{(\varrho)*}$, or $[\tau_1\tau_2]$. That is: if we cut a photon line open in K_0, we get $K(t_1t_2)$, which defines a real transition. This suffices to show that kernels can be connected in a simple way to variational derivatives of type (59), independently of perturbation theory, the resulting equations being of the branching type.

This argument is, allegedly, very incomplete; it is clear, however, that it can be made as general and consistent as desired.

A quite different way for obtaining equations of the branching type ensues from the application, after suitable generalizations, of prescriptions given by FERRETTI (9); these require only the existence of λ-derivatives up to the order considered. We do not insist on this argument any longer, since we deem the procedure adopted in the present work to be well suited for a manageable and complete mathematical formulation of the theory.

It may be well to remark explicitly that, besides the integral equations written by us, also other equations, integro-differential with respect to the space-time variables, are readily obtainable by operating on (30) or on the branching equations with the Dirac and d'Alembert operators; we forego recording them here, since their derivation is quite trivial. Multilinear equations among the kernels can also be deduced: for instance, bilinear from the unitary condition; although their consideration may be of interest in particular problems, we again forego entering into any details, all relevant information being already contained in the linear branching equations.

4·5. – We consider in this section some of the relations among purely fermionic kernels K_n (writing now n for N_0) which follow from eq'.s (51), (53) and (55). Differentiating (51) with respect to λ^2 we find, remembering (55):

$$(61)\qquad \frac{\mathrm{d}}{\mathrm{d}(\lambda^2)} K\begin{pmatrix} x_1 x_2 \dots x_n \\ y_1 y_2 \dots y_n \end{pmatrix} =$$

$$= \frac{1}{2}\sum_{h=1}^{n} (-1)^{h-1}(x_h y_1) \iint \sum_{1,2} \mathrm{d}\xi_1 \mathrm{d}\xi_2 \gamma^1\gamma^2[\xi_1\xi_2] K\begin{pmatrix} x_1 \dots x_{h-1} x_{h+1} \dots x_n \xi_1\xi_2 \\ y_2 \dots y_h \quad y_{h-1} \dots y_n \xi_1\xi_2 \end{pmatrix} -$$

$$- \frac{\mathrm{d}}{\mathrm{d}(\lambda^2)}\left[\lambda^2 \iint \sum_{1,2} \mathrm{d}\xi_1 \mathrm{d}\xi_2 \gamma^1\gamma^2(x_1\xi_1)[\xi_1\xi_2] K\begin{pmatrix} \xi_1 x_2 \dots x_n \xi_2 \\ y_1 y_2 \dots y_n \xi_2 \end{pmatrix}\right],$$

which connects K_n with K_{n+1} only. It is now convenient to write (61) with a shortened operator notation:

$$(62)\qquad \frac{\mathrm{d}}{\mathrm{d}(\lambda^2)} K_n = \left(\boldsymbol{a} + \lambda^2 \boldsymbol{b}\, \frac{\mathrm{d}}{\mathrm{d}(\lambda^2)}\right) K_{n+1},$$

(9) B. FERRETTI: *Nuovo Cimento*, **10**, 1179 (1953).

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are immediately desumed from (61) to be integral operators independent of λ^2: likewise, write (55) as

$$\frac{\mathrm{d}}{\mathrm{d}(\lambda^2)} K_n = \boldsymbol{c} K_{n+2} . \tag{63}$$

Then, differentiating (62) and using (63), we find:

$$\boldsymbol{c} \frac{\mathrm{d}}{\mathrm{d}(\lambda^2)} K_n = \left[\boldsymbol{b} \left(\boldsymbol{a} + \lambda^2 \boldsymbol{b} \frac{\mathrm{d}}{\mathrm{d}(\lambda^2)} \right) + \left(\boldsymbol{a} + \lambda^2 \boldsymbol{b} \frac{\mathrm{d}}{\mathrm{d}(\lambda^2)} \right)^2 \right] K_n . \tag{64}$$

It would be immediate, but not worth our while at present, to give explicit form to (64). Although rather forbidding in appearence, this relation, which operates upon a *single* kernel, deserves perhaps a further study.

5. – Approximation Methods.

5·1. – The conclusion of the preceding discussion is that the branching equations can be taken as the *defining* equations of a field theory; the extension of our results to the current meson theories requires only trivial changes in notation and in the definitions of the functions S^F and D^F; the study of a four-fermion theory can be made, likewise, along the same lines. This fulfils our program of casting the theory into a mathematical shape adequate for a quantitative, rigorous investigation. The major problem to be dealt with next is, of course, that of renormalization; we propose to dedicate to it the sequel to this paper. It is hoped that a formulation of this problem may be achieved in which, again, physics may intervene only in one initial stage, mathematics doing all the rest automatically.

We have taken the lead, in this work, from some formal resemblance existing between the expansions (30) and those familiar in the Fredholm theory. We would rather stress here, however, that the quantitative differences between, say, $K_{N_0,P_0}/K_0$ (we consider, in this Part, for the sake of simplicity, only the case that no external fields are present) as obtained from (30) and a corresponding Fredholm expansion seem far more substantial than this resemblance. To see this clearly, assume that we are dealing with a theory which has been rendered convergent artificially, e.g. by means of suitable re-definitions of the functions S^F and D^F. Then, while the standard numerator and denominator of the Fredholm theory are integral functions of λ, no such conclusion can be reached, at least on a first inspection, in our case: the usual tests cannot be applied, the only conclusion being that, even in a convergent two-field theory, there is no *a priori* reason to expect the radius of convergence to be

infinite. A deeper investigation of the subject must needs await a thorough treatment of renormalization.

We show in the next section how the branching equations can be made to yield a variety of approximation techniques. Our treatment of the question is, intentionally, very sketchy and incomplete, since renormalization will bring about substantial changes; it will not alter, however, our qualitative conclusions.

5·2. – We consider only the equations (51) or (52) for purely fermionic kernels K_n; if also the equations containing λ-derivatives are taken into account, the number of possible methods increases considerably. As we have said, we ignore throughout renormalization, and so, for instance, the fact that, after renormalization, $K\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ becomes simply $(x_1 y_1)$. The available renormalization techniques seem, however, applicable without changes, if desired.

The quantity of interest is, in our case, K_n; we can therefore use only the eq.'s (51). One possible approach is the following: the typical equation (51) connects K_r with K_{r-1} and K_{r+1}; neglect in it K_{r+1} to get $K_r^{(1)}$: one immediately finds $K_n^{(1)} = \begin{pmatrix} x_1 x_2 \dots x_n \\ y_1 y_2 \dots y_n \end{pmatrix}$. Then evaluate $K_r^{(2)}$ by replacing the K_{r+1} neglected before with $K_{r+1}^{(1)}$; and so on. One has only to evaluate (divergent, renormalizable) integrals: this is, clearly, the perturbative method; the difference from (30) is that, in (30), K_0 is badly entangled with the significant terms, while now it is neatly factored out. The convergence of the successive approximations can be tested, at least in principle; this is indeed the way to follow for such a test—but for the cumbersomeness of the known renormalization procedures.

Another possible approach is the following: into each of the equations (51) which give K_r in terms of K_{r-1} and K_{r+1}, substitute the value of K_{r+1} as given by the next equation. The result is a set of integral equations for the unknown functions K_r, where we take K_{r-1} and K_{r+2} to be the « known terms ». We can approximate K_n by neglecting, first, all terms K_{r+2}: this gives $K_1^{(1)}$, then $K_2^{(1)}$, then $K_3^{(1)}$, etc., as solutions of an integral equation (whose nature remains the same throughout the procedure). Obtain then the second approximation by means of the same step-wise process, after replacing the terms K_{r+2} neglected before with $K_{r+2}^{(1)}$; and so on. This method, which is discussed below in greater detail, appears akin to the one due to TAMM and DANCOFF, as is seen from eq. (48) read from right to left; there is now, however, the possibility of estimating the convergence of the successive approximations in a simple way, and one has to deal with only one type of equation, at all stages.

In applying either of these procedures, there is still, as we shall see, a considerable freedom of choice in the way of arriving from K_1 at K_n. Both

may be used at a time, in any wished manner, provided it be well defined; not to speak of the possibility of using the equations with λ-derivatives, as a check or as equations to solve. This shows that the only trouble, perhaps, now is that we have *too many* methods of approximating the (assumed existing) solutions; a comparative study of their merits must follow, however, that of renormalization.

We conclude by showing to which type of integral equation the second of the methods proposed above leads. It will suffice to consider the case of K_2. We have:

$$K\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = K_0(x_1 y_1) - \lambda^2 \iint \sum_{1,2} \mathrm{d}\xi_1 \, \mathrm{d}\xi_2 \gamma^1 \gamma^2 (x_1 \xi_1)[\xi_1 \xi_2] K\begin{pmatrix} \xi_1 \xi_2 \\ y_1 \xi_2 \end{pmatrix}, \tag{65}$$

$$K\begin{pmatrix} x_1 x_2 \\ y_1 y_2 \end{pmatrix} = (x_1 y_1) K\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} - (x_1 y_2) K\begin{pmatrix} x_2 \\ y_1 \end{pmatrix} - \lambda^2 \iint \sum_{1,2} \mathrm{d}\xi_1 \, \mathrm{d}\xi_2 \gamma^1 \gamma^2 (x_1 \xi_1)[\xi_1 \xi_2] K\begin{pmatrix} \xi_1 x_2 \xi_2 \\ y_1 y_2 \xi_2 \end{pmatrix}, \tag{66}$$

$$K\begin{pmatrix} x_1 x_2 x_3 \\ y_1 y_2 y_3 \end{pmatrix} = (x_1 y_1) K\begin{pmatrix} x_2 x_3 \\ y_2 y_3 \end{pmatrix} - (x_1 y_2) K\begin{pmatrix} x_2 x_3 \\ y_1 y_3 \end{pmatrix} + (x_1 y_3) K\begin{pmatrix} x_2 x_3 \\ y_1 y_2 \end{pmatrix} - \lambda^2 \iint \sum_{1,2} \mathrm{d}\xi_1 \, \mathrm{d}\xi_2 \gamma^1 \gamma^2 (x_1 \xi_1)[\xi_1 \xi_2] K\begin{pmatrix} \xi_1 x_2 x_3 \xi_2 \\ y_1 y_2 y_3 \xi_2 \end{pmatrix}. \tag{67}$$

We expand now $K\begin{pmatrix} \xi_1 \xi_2 \\ y_1 \xi_2 \end{pmatrix}$ in (65) by ξ_2, say in the first row, to exploit the fact that $(\xi_2 \xi_2) = 0$; likewise in the other equations. The resulting equations for the first approximation are:

$$K^{(1)}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = K_0(x_1 y_1) + \lambda^2 \iint \sum_{1,2} \mathrm{d}\xi_1 \, \mathrm{d}\xi_2 \gamma^1 \gamma^2 (x_1 \xi_1)[\xi_1 \xi_2] K^{(1)}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \tag{68}$$

$$K^{(1)}\begin{pmatrix} x_1 x_2 \\ y_1 y_2 \end{pmatrix} = (x_1 y_1) K^{(1)}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} - (x_1 y_2) K^{(1)}\begin{pmatrix} x_2 \\ y_1 \end{pmatrix} + \lambda^2 \iint \sum_{1,2} \mathrm{d}\xi_1 \mathrm{d}\xi_2 \gamma^1 \gamma^2 (x_1 \xi_1)[\xi_1 \xi_2] \left\{ (\xi_2 y_1) K^{(1)}\begin{pmatrix} \xi_1 x_2 \\ \xi_2 y_2 \end{pmatrix} - (\xi_2 y_2) K^{(1)}\begin{pmatrix} \xi_1 x_2 \\ \xi_2 y_1 \end{pmatrix} \right\}. \tag{69}$$

. . . . . .

To get the second approximation, one might, at this point, evaluate simply $K_1^{(2)}$ by substituting (69) into (65); better, also add to the known term of (69) (written for $K_2^{(2)}$) the contribution of $K_3^{(1)}$. Or apply consistently the procedure outlined above; or combine this with the perturbation approach, by taking, in the first approximation (69) to (66), $\begin{pmatrix} x_1 x_2 \\ y_1 y_2 \end{pmatrix}$ as the known term; etc.

Although each way one may wish to follow constitutes a different method, with a different rapidity of convergence (if any), the corner stone of all such approaches is always an equation of type (69). This is not a Fredholm equation; its manipulation by means of methods other than the perturbative does not appear difficult, but it would be pointless to discuss it in a non-renormalized theory. All this we differ to a later time.

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APPENDIX I

We calculate the electromagnetic contribution to $M_{FI}^{(N)}$ without any restriction on the initial and final states. The work done in I must be modified, because: *a*) final photon states may coincide with initial photon states; *b*) final and/or initial photon states may be identical. Start with assuming $N \geqslant P_0$; with the notation of I, this means that terms other than $\sum : \Pi^{(2Q)} :$ in the expansion (I.9) may give a contribution $\neq 0$ ($N = 2Q + P_0$). The total electromagnetic contribution is readily seen to be:

$$\text{(AI.1)} \qquad \langle \Psi_F^{(b)} | T(\Pi) | \Psi_I^{(b)} \rangle = \sum_{\varrho=0}^{(P_0)} \sum_{c\binom{N}{2Q+2\varrho}} [h_1 h_2 \dots h_{2Q+2\varrho}] \langle \Psi_F^{(b)} | : j_1 \dots j_{P_0-2\varrho} : | \Psi_I^{(b)} \rangle ,$$

where $(P_0) = [P_0/2]$ is the max. integer contained in $P_0/2$ and N and P_0 have the same parity (otherwise $M^{(N)} \equiv 0$). $\sum_{c\binom{N}{2Q+2\varrho}}$ is $\sum$ over all the $\binom{N}{2Q+2\varrho}$ combination of N into $2Q+2\varrho$ places. Writing:

$$j_\alpha = A_{j\alpha} + t\bar{A}_{j\alpha} \qquad ((t \equiv 1;\ A_{j\alpha} = A_{\mu_{j\alpha}}(x_{j\alpha}))$$

$$: j_1 \dots j_{P_0-2\varrho} : = \sum_{k=0}^{P_0-2\varrho} t^k C_k ,$$

one sees that the expectation value of $: j_1 \dots j_{P_0-2\varrho} :$ may be $\neq 0$ only if $b - k = a - P_0 + 2\varrho + k$, i.e. $k = b - \varrho$ and $P_0 - 2\varrho - k = a - \varrho$, so that:

$$\text{(AI.2)} \qquad \langle \Psi_F^{(b)} : j_1 \dots j_{P_0-2\varrho} : \Psi_I^{(b)} \rangle = \langle \Psi_F^{(b)} | \sum_{c\binom{P_0-2\varrho}{b-\varrho}} \bar{A}_{j'_1} \dots \bar{A}_{j'_{b-\varrho}} A_{j''_1} \dots A_{j''_{a-\varrho}} | \Psi_I^{(b)} \rangle ,$$

where $\sum_{c\binom{P_0-2\varrho}{b-\varrho}}$ is over all combinations $j'_1 \dots j'_{b-\varrho} j''_1 \dots j''_{a-\varrho}$ of $j_1 \dots j_{P_0-2\varrho}$ into $\binom{P_0-2\varrho}{b-\varrho} = \binom{P_0-2\varrho}{a-\varrho}$ places. We account now for the possibility that initial and/or final photon states may coincide by assuming, quite in general, that, out of the a initial states, τ_1 coincide with state 1, τ_2 coincide with state $2 \neq 1, \dots, \tau_\alpha$ with state α, with

$$\tau_1 + \tau_2 + \dots + \tau_\alpha = a \; ; \qquad |\Psi_I^{(b)}\rangle = |\Psi(\tau_1, \tau_2, \dots, \tau_\alpha)\rangle \; ;$$

likewise, out of the b final states, $\sigma_{1'}$ coincide with state $1'$, $\sigma_{2'}$ with state $2'$, ..., $\sigma_{\beta'}$ with state β', with:

$$\sigma_{1'} + \sigma_{2'} + \dots + \sigma_{\beta'} = b \; ; \qquad |\Psi_F^{(b)}\rangle = |\Psi(\sigma_{1'}, \sigma_{2'}, \dots, \sigma_{\beta'})\rangle \; .$$

The only contribution to (AI.2) comes then from the parts

$$\sum_{\lambda=1}^{\alpha} z^{(\lambda)}_{\mu_{j''}}(x_{j''}) a_\lambda \qquad \text{of} \qquad A_{j''} \; ,$$

and

$$\sum_{\lambda'=1'}^{\beta'} \bar{z}^{(\lambda')}_{\mu_{j'}}(x_{j'}) a^*_{\lambda'} \qquad \text{of} \qquad \bar{A}_{j'} \equiv A^*_{j'} \qquad (\bar{z} \equiv z^*) \; ,$$

where, of course, each a_λ, $a^*_{\lambda'}$ appears only once. Simple algebra shows that

$$\bar{A}_{j'_1} \dots \bar{A}_{j'_{b-\varrho}} A_{j''_1} \dots A_{j''_{a-\varrho}} = \sum_{f,g} f_{1'}! \dots f_{\beta'}! \, g_1! \dots g_\alpha! \cdot$$

$$\cdot \begin{bmatrix} \bar{z}^{(1')} \dots \bar{z}^{(1')} & \bar{z}^{(2')} & \dots \bar{z}^{(2')} & \dots \bar{z}^{(\beta')} \dots \bar{z}^{(\beta')} \\ j'_1 \dots j'_{f_{1'}} & j'_{f_{1'}+1} & \dots j'_{f_{1'}+f_{2'}} \dots \dots \dots j'_{b-\varrho} \end{bmatrix} \cdot \begin{bmatrix} z^{(1)} \dots z^{(1)} & z^{(2)} & \dots z^{(2)} & \dots z^{(\alpha)} \dots z^{(\alpha)} \\ j''_1 \dots j''_{g_1} & j''_{g_1+1} & \dots j''_{g_1+g_2} \dots \dots \dots j''_{a-\varrho} \end{bmatrix} \cdot$$

$$\cdot (a^*_{1'})^{f_{1'}} \dots (a^*_{\beta'})^{f_{\beta'}} \cdot (a_1)^{g_1} \dots (a_\alpha)^{g_\alpha} \; ,$$

where $\sum_{f,g}$ is over all sets of f, g such that $f_{1'} + f_{2'} + \dots + f_{\beta'} = b - \varrho$ and $g_1 + g_2 + \dots + g_\alpha = a - \varrho$, the permanents being defined as in I. (AI.2) becomes thus:

$$\text{(AI.3)} \qquad \langle \Psi_F^{(b)} | : j_1 \dots j_{P_0-2\varrho} : | \Psi_I^{(b)} \rangle = \sum_{f,g} \frac{1}{f_{1'}! \dots f_{\beta'}! \, g_1! \dots g_\alpha!} \cdot$$

$$\cdot \begin{bmatrix} \bar{z}^{(1')} \dots \bar{z}^{(1')} \dots \bar{z}^{(\beta')} \dots \bar{z}^{(\beta')} & z^{(1)} \dots z^{(1)} \dots z^{(\alpha)} \dots z^{(\alpha)} \\ j_1 \dots j_{f_{1'}} \dots j_{b-\varrho} \dots \dots & \dots \dots \dots \dots \dots \dots \dots j_{P_0-2\varrho} \end{bmatrix} \cdot \langle \Psi_F^{(b)} \; (a^*_{1'})^{f_{1'}} \dots (a^*_{\beta'})^{f_{\beta'}} (a_1)^{g_1} \dots (a_\alpha)^{g_\alpha} | \Psi_I^{(b)}$$

But: $(a_1)^{g_1}|\Psi(\tau_1, ..., \tau_\alpha)\rangle = (a_1)^{g_1-1}\sqrt{\tau_1}\,|\Psi(\tau_1-1, ..., \tau_\alpha)\rangle$, etc., so that:

$$\text{(AI.4)}\qquad \langle\Psi_F^{(b)}|:j_1 \dots j_{P_0-2Q}:|\Psi_I^{(b)}\rangle = \sum_{f,g}{}' \frac{1}{f_{1'}!\dots f_{\beta'}!\,g_1!\dots g_\alpha!}\begin{bmatrix} z^{(\lambda_1)} & \dots & z^{(\lambda_{P_0-2Q})} \\ j_1 & \dots & j_{P_0-2Q}\end{bmatrix}\cdot$$

$$\cdot\frac{\sqrt{\sigma_{1'}!\dots\sigma_{\beta'}!}\,\sqrt{\tau_1!\dots\tau_\alpha!}}{\sqrt{(\sigma_{1'}-f_{1'})!\dots(\sigma_{\beta'}-f_{\beta'})!}\cdot\sqrt{(\tau_1-g_1)!\dots(\tau_\alpha-g_\alpha)!}}\cdot$$

$$\cdot\langle\Psi(\sigma_{1'}-f_{1'}, ..., \sigma_{\beta'}-f_{\beta'})|\Psi(\tau_1-g_1, ..., \tau_\alpha-g_\alpha)\rangle\,.$$

Here: the prime affixed to $\sum_{f,g}{}'$ denotes the additional limitations $\sigma_h - f_h \geqslant 0$, $\tau_k - g_k \geqslant 0$; $z^{(a+1)}, ..., z^{(P_0)}$ are related to $\bar{z}$ z^* by (4); $f_{h'}!$ denotes that among the z describing final photons there are $f_{h'} \to$ photons in state h', etc. The expectation value can be given a simple expression, on introducing the symbols

$$[z^{(h)}z^{(k)}] = [z^{(k)}z^{(h)}] = \begin{cases} 0 \\ 0 \\ \bar{\delta}_{hk}\end{cases}$$

as defined in (4) in the text. It can then be verified that:

$$\text{(AI.5)}\qquad \langle\Psi(\sigma_{1'}-f_{1'}, ..., \sigma_{\beta'}-f_{\beta'})|\Psi(\tau_1-g_1, ..., \tau_\alpha-g_\alpha)\rangle =$$

$$= \frac{1}{\sqrt{(\sigma_{1'}-f_{1'})!\dots(\sigma_{\beta'}-f_{\beta'})!\,(\tau_1-g_1)!\dots(\tau_\alpha-g_\alpha)!}}\,[z^{(\lambda_{P_0-2Q+1})}\dots z^{(\lambda_{P_0})}]\,.$$

The right side of this expression is indeed $\neq 0$ only if

$$(\sigma_{1'}-f_{1'})!\dots(\sigma_{\beta'}-f_{\beta'})! = (\tau_1-g_1)!\dots(\tau_\alpha-g_\alpha)!$$

and, then, if all final states coincide, to within the order, with initial states of the same population. In such case, a simple calculation shows that its value is 1. (We think that this justification may suffice; this result has been derived, actually, in a more direct and pedestrian way). From (AI.1, 4, 5) we find:

$$\text{(AI.6)}\qquad \langle\Psi_F^{(b)}|\,T(\Pi)\,|\Psi_I^{(b)}\rangle = \frac{1}{\sqrt{\sigma_{1'}!\dots\sigma_{\beta'}!\,\tau_1!\dots\tau_\alpha!}}\cdot\sum_{\varrho=0}^{(P_0)}\sum_{C(2Q+2\varrho)}^{N}[h_1\dots h_{2Q+2\varrho}]\cdot$$

$$\cdot\sum_{f,g}{}'\binom{\sigma_{1'}}{f_{1'}}\dots\binom{\sigma_{\beta'}}{f_{\beta'}}\binom{\tau_1}{g_1}\dots\binom{\tau_\alpha}{g_\alpha}\cdot\begin{bmatrix} z^{(\lambda_1)} & \dots & z^{(\lambda_{P_0-2Q})} \\ j_1 & \dots & j_{P_0-2}\end{bmatrix}[z^{(\lambda_{P_0-2Q+1})}\dots z^{(\lambda_{P_0})}]\,.$$

Consider now the expression

$$\text{(AI.7)}\qquad \frac{1}{\sqrt{\sigma_{1'}!\sigma_{2'}!\dots\sigma_{\beta'}!\,\tau_1!\,\tau_2!\dots\tau_\alpha!}}[1\,2\dots N z^{(1)}\dots z^{(P_0)}] = \frac{1}{\sqrt{\sigma_{1'}!\dots\sigma_{\beta'}!\,\tau_1!\dots\tau_\alpha!}}\cdot$$

$$\cdot\sum_{\varrho=0}^{(P_0)}\sum_{C(2Q+2\varrho)}^{N}[h_1\dots h_{2Q+2\varrho}]\cdot\sum_{C(2Q)}^{(P_0)}\begin{bmatrix} z^{(\lambda_1)} & \dots & z^{(\lambda_{P_0-2Q})} \\ j_1 & \dots & j_{P_0-2Q}\end{bmatrix}\cdot[z^{(\lambda_{P_0-2Q+1})}\dots z^{(\lambda_{P_0})}]\,.$$

This expansion is an immediate consequence of the theorem expressed by formula (17) in A, after its trivial extension to the case of hafnians and the removal of the non significant restriction to determinants, or permanents, of even order only. (AI.7) is expanded without taking care of the fact that $\tau_1, \ldots, \tau_\alpha$ initial photons are in the same state, etc. If this fact is taken into account, it reduces to (AI.6). Therefore, in the most general case:

$$\text{(AI.8)} \qquad \langle \Psi_F^{(b)} | T(\Pi) | \Psi_I^{(b)} \rangle = \frac{1}{\sqrt{\sigma_{1'}! \sigma_{2'}! \ldots \sigma_{\beta'}! \tau_1! \tau_2! \ldots \tau_\alpha!}} \cdot \cdot [\xi_1 \xi_2 \ldots \xi_N z^{(1)} z^{(2)} \ldots z^{(P_0)}],$$

to be expanded with the rules given in the text. The same result can be verified to hold also if $N < P_0$ (and has the same parity, or else $M_{FI}^{(N)} = 0$).

Appendix II

Using the notation of I, the total contribution of the electron-positron field to M_{FI}^{N}, when final states are allowed to coincide with initial states, is given by:

$$\text{(AII.1)} \qquad \langle \Psi_F^{(f)} | T(P) | \Psi_I^{(f)} \rangle = \sum_{\varrho=0}^{N_0} \sum_{C\binom{2N}{2M+2\varrho}} (h_1 \ldots h_{2M+2\varrho})' \langle \Psi_F^{(f)} | : j_1 \ldots j_{2N-2\varrho} : | \Psi_I^{(f)} \rangle ;$$

(I.23) reads now

$$\text{(AII.2)} \qquad (h_1 \ldots h_{2M+2\varrho})' = (-1)^{\left(M+\frac{\varrho}{2}\right)+\tau_\varrho'} \cdot \begin{vmatrix} a_{k_1' k_{h_1}'} & \ldots & a_{k_1' h_{M+\varrho}'} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{k_{M+\varrho}' h_1'} & \ldots & a_{k_{M+\varrho}' h_{M+\varrho}'} \end{vmatrix}$$

where τ_ϱ' is the parity of $k_1' \ldots k_{M+\varrho}' h_1' \ldots h_{M+\varrho}'$ with respect to $h_1 \ldots h_{2M+2\varrho}$ (the k's are all the odd, the h's all the even indices ordered as they appear in $h_1, \ldots, h_{2M+2\varrho}$). We depart now from the procedure followed in I and set

$$J_\varrho = : j_1 \ldots j_{2N_0-2\varrho} : = (-1)^{\tau_\varrho''} \cdot J_\varrho' ,$$

with

$$J_\varrho' = : k_1'' \ldots k_{N_0-\varrho}'' h_1'' \ldots h_{N_0-\varrho}'' : ,$$

where τ_ϱ'' is the parity of $k_1'' \ldots k_{N_0-\varrho}'' h_1'' \ldots h_{N_0-\varrho}''$ (that is, with all the odd indices k shifted, in their order, to the left) with respect to $j_1 \ldots j_{2N_0-2\varrho}$.

But:

$$k'' = \bar{u}_{\alpha_{(k'')}}(x_{(k'')}) + s v_{\alpha_{(k'')}}(x_{(k'')}) \equiv \bar{u}_{k''} + s v_{k''} \qquad (s = 1)$$

$$h'' = u_{\beta_{(h'')}}(x_{(h'')}) + t v_{\alpha_{(h'')}}(x_{(h'')}) \equiv u_{\bar{h}''} + t \bar{v}_{h''} \qquad (t = 1)$$

so that:

$$J'_\varrho = \sum_{\mu,\lambda}^{0\dots N_0-\varrho} s^\mu t^\lambda : A_\mu B_\lambda : ,$$

where A_μ is a product of $\bar{u}$ and v, B_λ of u and $\bar{v}$. It is then quite simple to see that:

$$\text{(AII.3)}\quad J_\varrho = (-1)^{\tau''_\varrho} \sum_{\mu,\lambda}^{0\dots N_0-\varrho} s^\mu t^\lambda \sum_{C\binom{N_0-\varrho}{\mu}} \sum_{C\binom{N_0-\varrho}{\lambda}} (-1)^{p_\varkappa + p_\eta + \lambda(\mu+\sigma+\nu)+\mu\sigma} \cdot$$

$$\cdot \bar{v}_{\eta'_1} \dots \bar{v}_{\eta'_\lambda} \cdot \bar{u}_{\varkappa_1} \dots \bar{u}_{\varkappa_\nu} \cdot u_{\eta_1} \dots u_{\eta_\sigma} \cdot v_{\varkappa'_1} \dots v_{\varkappa'_\mu} ,$$

where $\nu + \mu = \sigma + \lambda = N_0 - \varrho$; $\sum_{C\binom{N_0-\varrho}{\mu}}$ is extended over all the combinations $\varkappa_1 \dots \varkappa_\nu \varkappa'_1 \dots \varkappa'_\mu$ of $k''_1 \dots k''_{N_0-\varrho}$ (of parity $p_\varkappa$) into μ places; $\sum_{C\binom{N_0-\varrho}{\lambda}}$ over all the combinations $\eta_1 \dots \eta_\sigma \eta'_1 \dots \eta'_\lambda$ (of parity p_η) of $h''_1 \dots h''_{N_0-\varrho}$ into λ places. Next, on expanding all the field operators into their creation and destruction parts, one finds that:

$$\text{(AII.4)}\quad \langle \Psi_F^{(f)} | \bar{v}_{\eta'_1} \dots v_{\varkappa'_\mu} | \Psi_I^{(f)} \rangle = \sum_{C\binom{q}{\lambda}} \sum_{C\binom{p}{\nu}} \sum_{C\binom{n}{\sigma}} \sum_{C\binom{m}{\mu}} \cdot$$

$$\cdot \begin{pmatrix} \bar{v}^{\tau'_1} \dots \bar{v}^{\tau'_\lambda} \\ \eta'_1 \dots \eta'_\lambda \end{pmatrix}' \begin{pmatrix} \bar{u}^{\sigma'_1} \dots \bar{u}^{\sigma'_\nu} \\ \varkappa_1 \dots \varkappa_\nu \end{pmatrix}' \begin{pmatrix} u^{\xi_1} \dots u^{\xi_\sigma} \\ \eta_1 \dots \eta_\sigma \end{pmatrix}' \begin{pmatrix} v^{\varepsilon_1} \dots v^{\varepsilon_\mu} \\ \varkappa'_1 \dots \varkappa'_\mu \end{pmatrix}' \cdot$$

$$\cdot \langle \Psi_F^{(f)} | b^*_{\tau'_1} \dots b^*_{\tau'_\lambda} a^*_{\sigma'_1} \dots a^*_{\sigma'_\nu} a_{\zeta_1} \dots a_{\zeta_\sigma} b_{\varepsilon_1} \dots b_{\varepsilon_\mu} | \Psi_I^{(f)} \rangle ,$$

where $\tau'_1 < \tau'_2 < \dots < \tau'_\lambda$ is any one of the $\binom{q}{\lambda}$ combinations of $1' \dots q'$, $\sum_{C\binom{q}{\lambda}}$ denotes sum over all of them, and the operators $b^*_{\tau'_1} \dots b^*_{\tau'_\lambda}$ are those which correspond to the final positrons. All the other symbols have a similar definition and meaning. We can collect now (AII.1, 2, 3, 4) and perform the two sums $\sum_{C\binom{N_0-\varrho}{\mu}}$ and $\sum_{C\binom{N_0-\varrho}{\lambda}}$, obtaining:

$$\text{(AII.5)}\quad \langle \Psi_F^{(f)} | T(P) | \Psi_I^{(f)} \rangle = \sum_{\varrho=0}^{N_0} \sum_{C\binom{2N}{2M+2\varrho}} \cdot (-1)^{\binom{M+\varrho}{2}+\tau'_\varrho+\tau''_\varrho} \cdot$$

$$\cdot \begin{vmatrix} a_{k'_1 h'_1} & \dots & a_{k'_1 h'_{M+\varrho}} \\ \dots & \dots & \dots \\ a_{k'_{M+\varrho} h'_1} & \dots & a_{k'_{M+\varrho} h'_{M+\varrho}} \end{vmatrix} \cdot \sum_{\mu,\lambda} (-1)^{\lambda(\mu+\sigma+\nu)+\mu\sigma} \cdot \sum_{C\binom{q}{\lambda}} \sum_{C\binom{p}{\nu}} \sum_{C\binom{n}{\sigma}} \sum_{C\binom{m}{\mu}} \begin{pmatrix} \bar{u}^{\sigma'_1} \dots \bar{u}^{\sigma'_\nu} v^{\varepsilon_1} \dots v^{\varepsilon_\mu} \\ k''_1 \dots \dots \dots k''_{N_0-\varrho} \end{pmatrix}' \cdot$$

$$\cdot \begin{pmatrix} u^{\xi_1} \dots u^{\xi_\sigma} \bar{v}^{\tau'_1} \dots \bar{v}^{\tau'_\lambda} \\ h''_1 \dots \dots \dots h''_{N_0-\varrho} \end{pmatrix} \cdot \langle \Psi_F^{(f)} | b^*_{\tau'_1} \dots b_{\varepsilon_\mu} | \Psi_I^{(f)} \rangle .$$

Setting, as in I and (5):

$$\bar{v}^{(1')} = u^{(n+1)}, \quad \dots, \quad \bar{v}^{(q')} = u^{(N_0)},$$
$$\bar{u}^{(1')} = v^{(m+1)}, \quad \dots, \quad \bar{u}^{(p')} = v^{(N_0)}$$

and writing

$$\beta_1 = \varepsilon_1, \quad \dots, \quad \beta_\mu = \varepsilon_\mu; \qquad \beta_{\mu+1} = m + \sigma'_1, \quad \dots, \quad \beta_{N_0-\varrho} = m + \sigma'_\nu;$$
$$\alpha_1 = \zeta_1, \quad \dots, \quad \alpha_\sigma = \zeta_\sigma; \qquad \alpha_{\quad +} = n + \tau'_1, \quad \dots, \quad \alpha_{N_0-\varrho} = n + \tau'_\lambda,$$

(AII.5) becomes, using the results of I:

$$\text{(AII.6)} \qquad \langle \Psi_F^{(f)} | T(P) \Psi_I^{(f)} \rangle = \sum_{\varrho=0}^{N_0} (-1)^{\left(N_0 - \frac{\varrho}{2}+1\right)} \sum_{\mu,\lambda} \sum_{C\binom{q}{\lambda}} \sum_{C\binom{p}{\nu}} \sum_{C\binom{n}{\sigma}} \sum_{C\binom{m}{\mu}} \cdot$$
$$\cdot (-1)^{\lambda(\mu+\sigma+\nu)+\mu\sigma+\mu\nu} \begin{pmatrix} 1 \dots N & v^{\beta_1} \dots v^{\beta_{N_0-\varrho}} \\ 1 \dots N & u^{\alpha_1} \dots u^{\alpha_{N_0-\varrho}} \end{pmatrix}_I' \cdot \langle \Psi_F^{(f)} | b^*_{\tau'_1} \dots b_{\varepsilon_\mu} | \Psi_I^{(i)} \rangle .$$

In the determinant appearing in (AII.6), the principal minor formed by the intersections of the u and v lines has vanishing elements, as was derived in I: this fact is reminded by the label I. Introducing the expressions of $\Psi_F^{(f)}$ and $\Psi_I^{(f)}$ one sees easily that

$$\langle \Psi_F^{(f)} | b^*_{\tau'_1} \dots b_{\varepsilon_\mu} | \Psi_I^{(f)} \rangle = (-1)^\alpha \langle 0 | a_{\sigma'_{\nu+1}} \dots a_{\sigma'_p} b_{\tau'_{\lambda+1}} \dots b_{\tau'_q} b^*_{\varepsilon_m} \dots b^*_{\varepsilon_{\mu+1}} a^*_{\zeta_n} \dots a^*_{\zeta_{\sigma+1}} | 0 \rangle ,$$

where

$$a = \sum_{h=1}^{\lambda} \tau'_h + \sum_{h=1}^{\nu} \sigma'_h + \sum_{h=1}^{\mu} \varepsilon_h + \sum_{h=1}^{\sigma} \zeta_h + \lambda q + \nu q + \nu p - \nu\lambda + \mu m + \binom{\mu}{2} +$$
$$+ \sigma m - \sigma\mu + \binom{\sigma}{2} + \sigma n .$$

We observe now that this expression can be $\neq 0$ only if $m - \mu = q - \lambda$, $n - \sigma = p - \nu$; it is given, adopting the definitions (6) introduced in the text, by (except for the factor $(-1)^\alpha$):

$$\text{(AII.7)} \qquad (-1)^{n-\sigma} \begin{pmatrix} \bar{u}^{\sigma'_{\nu+1}} \dots \bar{u}^{\sigma'_p} \\ u^{\zeta_{\sigma+1}} \dots u^{\zeta_n} \end{pmatrix} \cdot \begin{pmatrix} \bar{v}^{\tau'_{\lambda+1}} \dots \bar{v}^{\tau'_q} \\ v^{\varepsilon_{\mu+1}} \dots v^{\varepsilon_m} \end{pmatrix} =$$
$$= (-1)^{(q-\lambda+1)(n-\sigma)} \begin{pmatrix} v^{\varepsilon_{\mu+1}} \dots v^{\varepsilon_m} & \bar{u}^{\sigma'_{\nu+1}} \dots \bar{u}^{\sigma'_p} \\ u^{\zeta_{\sigma+1}} \dots u^{\zeta_n} & \bar{v}^{\tau'_{\lambda+1}} \dots \bar{v}^{\tau'_q} \end{pmatrix} = (-1)^{(q-\lambda+1)(n-\sigma)} \cdot \begin{pmatrix} v^{\beta'_1} \dots v^{\beta'_\varrho} \\ u^{\alpha'_1} \dots u^{\alpha'_\varrho} \end{pmatrix} .$$

We have thus, since $\sum_{\mu,\lambda} \sum_{C\binom{q}{\lambda}} \dots \sum_{C\binom{m}{\mu}}$ can be extended to $\sum_{C_\alpha\binom{N_0}{\varrho}} \sum_{C_\beta\binom{N_0}{\varrho}}$ over all

the combinations $\alpha_1 \dots \alpha_{N_0-\varrho}$, $\alpha'_1 \dots \alpha'_\varrho$ and $\beta_1 \dots \beta_{N_0-\varrho}$, $\beta'_1 \dots \beta'_\varrho$ of $1 \dots N_0$ (all new terms apparently introduced this way are $\equiv 0$), that:

$$\text{(AII.8)} \qquad \langle \Psi_F^{(f)} | T(P) | \Psi_I^{(f)} \rangle = \sum_{\varrho=0}^{N_0} \sum_{C_\alpha \binom{N_0}{\varrho}} \sum_{C_\beta \binom{N_0}{\varrho}} (-1)^{b + \sum_{h=1}^{N_0-\varrho} \alpha_h + \sum_{h=1}^{N_0-\varrho} \beta_h} \cdot \begin{pmatrix} 1 \dots N & v^{\beta_1} \dots v^{\beta_{N_0-\varrho}} \\ 1 \dots N & u^{\alpha_1} \dots u^{\alpha_{N_0-\varrho}} \end{pmatrix}' \begin{pmatrix} v^{\beta'_1} \dots v^{\beta'_\varrho} \\ u^{\alpha'_1} \dots u^{\alpha'_\varrho} \end{pmatrix}.$$

But, from the relations among λ, μ, ν, ..., it follows that:

$$b = \binom{N_0 - \varrho + 1}{2} + \lambda(\mu + \sigma + \nu) + \mu\sigma + \mu\nu + a + \nu m + \lambda n + \\ + (q - \lambda + 1)(n - \sigma) \approx p(m + n) + \binom{n}{2} - \binom{P}{2} = p_{FI} ,$$

which is the value already found in I in a different way. Finally:

$$\text{(AII.9)} \qquad \langle \Psi_F^{(f)} | T(P) | \Psi_I^{(f)} \rangle = (-1)^{p_{FI}} \begin{pmatrix} \xi_1 \xi_2 \dots \xi_N & v^{(1)} v^{(2)} \dots v^{(N_0)} \\ \xi_1 \xi_2 \dots \xi_N & u^{(1)} u^{(2)} \dots u^{(N_0)} \end{pmatrix}'$$

the determinant being now expanded with the rules given in the text, and not any more with those of I. The last formula is proved quite trivially, first by observing that each term of (AII.8) is a term of the expansion of (AII.9) with the correct sign, no two terms of (AII.8) coinciding, then by verifying that (AII.8) and (AII.9), after expansion, have the same number of terms, since:

$$\sum_{\varrho=0}^{N_0} \binom{N_0}{\varrho}^2 \binom{N}{N_0 - \varrho}^2 \varrho! [(N_0 - \varrho)!]^2 (N - N_0 + \varrho)! = (N + N_0)! .$$

Appendix III

We sketch here the proof of Furry's theorem for total graphs. Although well known considerations might spare us this trouble, we need this argument to prove formula (47) of the text. We start by proving that

$$\text{(46)} \qquad K(t_1 t_2 \dots t_{2h+1}) = 0 .$$

From (30) it follows that $K(t_1 t_2 \dots t_{2h+1})$ is a sum of integrals containing, in the N-th term, an integrand factor of the form:

$$\text{(AIII.1)} \qquad \sum_{(\text{all } \alpha,\ \beta)} \gamma^{\mu_1}_{\alpha_1 \beta_1} \dots \gamma^{\mu_N}_{\alpha_N \beta_N} \begin{pmatrix} \xi_1 \xi_2 \dots \xi_N \\ \xi_1 \xi_2 \dots \xi_N \end{pmatrix}, \qquad N \text{ odd}.$$

The theorem is proved by showing that (AIII.1) is $\equiv 0$ *prior to integration*, provided there are no external fields (we are talking about *total* graphs). Explicitly, (AIII.1) is given by:

$$\text{(AIII.2)}\quad \frac{1}{2^N}\sum \begin{vmatrix} 0 & S^F_{\beta_1\alpha_2}(\xi_1-\xi_2)\gamma^{\mu_2}_{\alpha_2\beta_2} & \dots & S^F_{\beta_1\alpha_N}(\xi_1-\xi_N)\gamma^{\mu_N}_{\alpha_N\beta_N} \\ \dots & \dots & \dots & \dots \\ S^F_{\beta_N\alpha_1}(\xi_N-\xi_1)\gamma^{\mu_1}_{\alpha_1\beta_1} & S^F_{\beta_N\alpha_2}(\xi_N-\xi_2)\gamma^{\mu_2}_{\alpha_2\beta_2} & \dots & 0 \end{vmatrix},$$

N *odd*.

Set $a_{hk} = (S^F(\xi_h - \xi_k)\gamma^{\mu_k})_{\beta_h\beta_k}$. In the expansion of det.$\|a_{hk}\|$ we can, in each term, arrange the factors so that their first and second indices form cyclic permutations: a term appears then as the product of factors of type:

$$a = a_{r_1r_2}a_{r_2r_3}\dots a_{r_{s-1}r_s}a_{r_sr_1},$$

and its sign depends only upon the sum of the orders of the cycles $(r_1r_2\dots r_{s-1}r_s)$ it contains (each term $\neq 0$ contains cycles of order 2 at least, since $a_{hh}=0$ by definition): this is the proper grouping to take spurs. Since N is odd, there is at least one cycle a of odd order in a given term $A=aB$. There is then certainly, in the expansion, another term $A'=a'B$, with

$$a' = a_{r_1r_s}a_{r_sr_{s-1}}\dots a_{r_2r_1},$$

that is, a' corresponds to the cycle $(r_sr_{s-1}\dots r_2r_1)$. Clearly, A and A' have the same sign; their sum contains therefore the factor

$$\mathrm{Sp}\,(a+a') \cong \mathrm{Sp}\,\{S^F(\xi_{r_1}-\xi_{r_2})\gamma^{\mu_{r_2}}\dots S^F(\xi_{r_s}-\xi_{r_1})\gamma^{\mu_{r_1}}\} + \\ + \mathrm{Sp}\,\{S^F(\xi_{r_1}-\xi_{r_s})\gamma^{\mu_{r_s}}\dots S^F(\xi_{r_2}-\xi_{r_1})\gamma^{\mu_{r_1}}\},$$

which is immediately seen to be zero. Indeed:

$$S^F = (\gamma^\mu\partial_\mu - m)\Delta^F, \qquad \Delta^F \text{ even};$$

each spur can yield only even powers of m, so that we may change m into $-m$ in the first, and write $\partial_{hk} = -\partial_{kh}$. The first spur becomes therefore the negative of the second (with all its terms in reverted order, which is irrelevant): their sum is zero.

A moment's thought shows that, by this argument, we have proved also formula (47), as was our purpose.

RIASSUNTO

Il calcolo di un elemento della matrice U tra stati iniziale e finale arbitrari viene ricondotto a quello di un « nucleo », la cui espressione, dipendente solo dal numero delle particelle coinvolte, è data, esplicitamente, come sviluppo perturbativo. Si dimostra che tali nuclei soddisfano sistemi di « equazioni di diramazione » che valgono indipendentemente da metodi perturbativi e possono essere prese come fondamento assiomatico della teoria, la covarianza relativistica risultando evidente. Si danno sistemi completi di equazioni siffatte per i nuclei e per le loro derivate rispetto all'intensità di interazione λ; ogni altra possibile equazione essendo necessariamente deducibile da quelle scritte. Tutti i nuclei corrispondenti a processi coinvolgenti bosoni reali sono ottenibili, con semplici integrazioni, da quelli relativi a processi tra soli fermioni; le equazioni di diramazione per questi ultimi, date anche esplicitamente, bastano a definire la teoria. Un nucleo e le sue derivate prima e seconda rispetto a λ soddisfano un'unica relazione integrale. Le equazioni di diramazione possono essere approssimate con una varietà di metodi, che, mentre estendono e generalizzano quelli noti, permettono sempre, almeno in linea di principio, lo studio della convergenza. Questioni di rinormalizzazione, esistenza di soluzioni, ecc., verranno studiate nel seguito a questo lavoro.

On the Interaction of μ-Mesons with Matter at High Energies.

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Summary. — The production of penetrating showers at depths of 50 and 200 metres water equivalent by high energy μ-mesons has been studied with a counter hodoscope. The values of the cross-section found (in lead) for events of this kind are consistent with those expected assuming an electromagnetic interaction mechanism, according to which the penetrating showers can be initiated by the photoproduction of a π-meson or by a high energy recoil nucleon. The agreement between the measured and calculated cross-section is very good, if it is assumed that the total photonuclear cross-section, $\sigma_{\gamma tot} = 3 \cdot 10^{-28}$ cm²/nucleon, remains unchanged even at the high energies concerned at the depths considered. Therefore it seems unnecessary to invoke a new interaction in order to explain the experimental results.

1. – Introduction.

With a counter hodoscope we have studied the production of penetrating showers by μ-mesons with the aim of determining the behaviour of the cross-section for such processes as a function of the average energy of the incident particles. The interest of measurements of this kind lies in its connection with the problem of the interaction of high energy μ-mesons with nuclei. In fact the information which can be drawn, concerns the mechanism of interaction between the electro-magnetic field of the incident μ-meson and the mesonic field of the nucleons and possibly other kinds of interaction which could be important at high energies.

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In previous papers (1) we have described our apparatus and discussed the results obtained at a depth of 50 m w.eq.. We present here a few results obtained in a new set of measurements performed in a tunnel at Tivoli (the « Castagnola » tunnel of the « Società Romana di Elettricità », at 400 m a.s.l.).

The overlying material is a compact chalky rock of average density of 2.49 g/cm^3; the corresponding depth is 200 m w.eq. from the top of the atmosphere. The average energy of the incident particles at this depth is $E_0 = 4.40 \cdot 10^{10}$ eV; while at 50 m w.eq. it was $E_0 = 1.14 \cdot 10^{10}$ eV.

The measurements were made from February to August 1953. The total time of observation was 53 081 minutes; the total path lenght crossed in lead by ionizing particles was 4 868 m. 10 040 photographs have been examined.

2. – Results and discussion.

The arrangement of counters and absorbers is shown in Fig. 1.

Among the many events which have been recorded, we shall consider in this paper only those which match the two following requirements and that we will call conventionally penetrating showers: 1) more than one counter is discharged in every one of the four trays B, C, D, E. 2) 4 of the discharged counters—one from each of the trays—must be colinear. The imposed conditions suggest a primary particle producing secondaries in Σ (5 cm Pb) by its interaction with matter.

At our depths the incident ionizing particles are almost all μ-mesons. Therefore we will call them in the following simply μ-mesons and we will give our experimental results in the form of the ratio between the number of penetrating showers and the number of incident μ-mesons. Such a ratio, indicated by P_{ex} is given in column 2 of Table I.

In order to understand the significance of these results we will consider in the following three different kinds of interaction with matter which may have produced hodoscope pictures of the type which we have observed:

a) An electron-photon shower can be started in Σ by a high energy μ-meson through the processes of collision with electrons, bremsstrahlung, and pair production. The energy transfer must be such that the shower can

(1) E. Amaldi, C. Castagnoli, A. Gigli and S. Sciuti: *Nuovo Cimento*, **9**, 453 (1953); **9**, 969 (1952); C. Castagnoli, A. Gigli and S. Sciuti: *Nuovo Cimento*, **10**, 893 (1953). We refer to these papers for matters concerning the criteria we have followed in the interpretation of the hodoscope pictures, the notations we have used to distinguish the different kinds of events from each other, and the physical conclusions we have drawn from the numerous types of events which were observed at the depth of 50 m w.eq.

go through 35 radiation units of lead in order to discharge the counters in tray E.

b) A penetrating shower can be generated in Σ by a high energy μ-meson through a process of interaction between the electromagnetic field of the incident meson and the nucleons. Fig. 1 illustrates an event which we have interpreted as a penetrating shower: a single penetrating particle, P_1, crosses the whole telescope; it is accompanied by one or more secondary penetrating particles which have been generated in Σ; one at least of these, P_2, penetrates all the interposed lead layers.

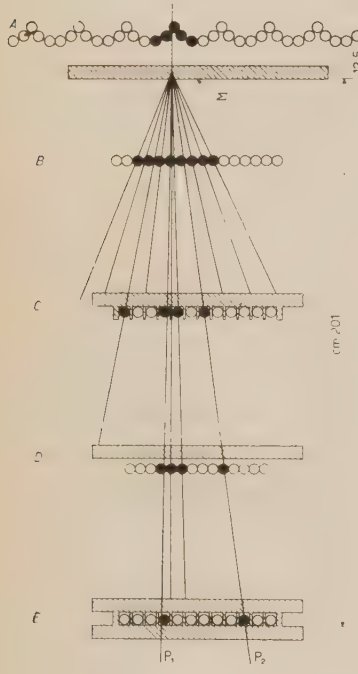

Fig. 1.

c) Finally we have to consider the possibility that a penetrating shower of the kind represented in Fig. 1 can be started by a specific interaction of μ-mesons with nucleons.

In order to evaluate the importance of process *a*) we have calculated the probability P_t that an electron-photon shower generated in Σ by a high energy μ-meson, discharges at least one counter of tray D and one counter of tray E. As the shower must be very energetic in order to reach tray E, it is certain that several of the counters in trays B and C will have been discharged. We have already described in some detail (1) the method of calculation, which is very similar to that suggested by GREISEN (2).

We have used an energy spectrum for the incident radiation of the type

$$N(E)\,\mathrm{d}E = \frac{2E_0^2}{(E + E_0)^3}\,\mathrm{d}E\,,$$

($E_0 = 1.14 \cdot 10^{10}$ eV at 50 m w.eq. and $E_0 = 4.40 \cdot 10^{10}$ eV at 200 m w.eq.).

In calculating the penetration of the electron-photon shower we have considered the contribution of low energy photons: as a low energy photon is necessarily absorbed when it is detected, we have taken into account the fact that the probability of detection of a shower in tray E is not indipendent of its detection in tray D.

(2) K. GREISEN: *Phys. Rev.*, **75**, 1071 (1949).

For the probability of detection of a low energy photon with a counter we have used the value 0.05. The results of the calculation are given in column 3 of Table I where they are compared with the experimental results at both depths.

TABLE I. – *Comparison between the experimental relative frequency of the events concerned* P_{ex} *and the expected,* P_t, *on the hypothesis discussed in the text.* (P_{ex} *– ratio between the number of events recorded and the number of mesons crossing the apparatus).*

Depth (m w. eq.)	P_{ex}	P_t	P_{ex}/P_t
1	2	3	4
50	$(8.40 \pm 1) \cdot 10^{-5}$	$8.00 \cdot 10^{-5}$	1.05
200	$(25.60 \pm 5) \cdot 10^{-5}$	$10.24 \cdot 10^{-5}$	2.50

It is not easy to draw a definite conclusion from the comparison between the calculated values and the experimental ones, because the uncertainties involved in any calculation on the electron-photon cascade are too great.

However the results given in Table I appear to indicate that, at 50 m w.eq. an appreciable fraction, or perhaps the majority of the events called penetrating showers may be purely electron-photon showers; but at 200 m w.eq. (where the average energy of the incident particles is ~ 4 times larger) it seems that a good fraction of the events is due to processes of a different kind.

In order to discuss the processes *b*) and *c*) we will assume in the following that none of the considered events is of a simple electron-photon nature but that all of them can be attributed to nuclear interactions of μ-mesons with nucleons. It is clear that under such an assumption we will establish an upper limit for the cross-section for production of penetrating showers.

The numerical values of the cross-sections obtained under this assumption at both depths, are given in Table II.

TABLE II. – *Cross-section for the production of events interpreted as penetrating showers crossing at least* 15 cm Pb.

Penetration of secondary particle (cm Pb)	50 m w.eq. $E_0 = 1.14 \cdot 10^{10}$ eV $\Sigma = 5$ cm Pb		200 m w.eq. $E_0 = 4.40 \cdot 10^{10}$ eV $\Sigma = 5$ cm Pb	
	Number of observed events	σ_μ (cm²/nucleon)	Number of observed events	σ_μ (cm²/nucleon)
1	2	3	4	5
$\geqslant 15$	39	$(2.4 \pm 0.4) \cdot 10^{-30}$	25	$(7.5 \pm 1.0) \cdot 10^{-30}$

In order to have some information about the relative importance of the two types of processes *b*) and *c*), we can now try to calculate the cross-section σ_μ of μ-mesons with nucleons for process *b*) and compare such a value with our experimental results.

Such a calculation can be done by following the semi-classical procedure of WEIZSÄCKER-WILLIAMS ([3]) in which the electromagnetic field of the incident μ-meson is resolved in an equivalent spectrum of photons which interact with the nucleons with a cross-section σ_γ. A π-meson which has been produced in the photonuclear interaction can be subsequently reabsorbed in the same nucleus, producing an excited nucleus which can give rise to a star with possibly several associated penetrating particles.

The spectrum of virtual photons which represent the electromagnetic field of the incident μ-meson of energy E is given by

$$N(\varepsilon)\,\mathrm{d}\varepsilon = (2/173\pi)\ln(E/\varepsilon)\,\mathrm{d}\varepsilon/\varepsilon \tag{1}$$

where ε is the energy of the photon; the relation between σ_μ and σ_γ is

$$\sigma_\mu = \int_{\varepsilon_m}^{E} \sigma_\gamma N(\varepsilon)\,\mathrm{d}\varepsilon\,, \tag{2}$$

where ε_m is the lowest energy that a photon must have in order to create a penetrating shower of the kind which we consider in the present paper.

The cross-section σ_γ for the photoproduction of a π-meson has been measured by a number of workers ([4]) with synchrotrons at 300 MeV; at this energy $\sigma_\gamma \cong 1\cdot10^{-28}$ cm^2/nucleon. On the other hand KECK ([5]) has shown, by measurements which also have been made with a synchrotron at 300 MeV, that the cross-section for photoproduction of single recoil nucleons of high energy is not negligible and that such processes are not always associated with the production of a star. As a result of this observation BARRET *et al.* ([6]) think it likely that the *total* cross-section for photonuclear processes at about 300 MeV must be $2\text{-}3\cdot10^{-28}$ cm^2/nucleon; at high energies the production of a single π-meson or of a high energy recoil nucleon could produce a nucleonic shower by interaction with the same nucleus or with other nuclei. It follows that

([3]) See for instance: E. P. GEORGE: *Progress in Cosmic Ray Physics* (New York, 1951), pp. 429-430.

([4]) See for instance: H. JAHN: *Kosmische Strahlung* (Herausgegeben von W. HEISENBERG; Berlin, 1953), pp. 289-296.

([5]) J. C. KECK: *Phys. Rev.*, **85**, 410 (1952).

([6]) P. H. BARRET, M. L. BOLLINGER, G. COCCONI, Y. HEISENBERG and K. GREISEN: *Rev. Mod. Phys.*, **24**, 133 (1952).

at high energies all photonuclear interactions can contribute to the formation of penetrating showers; therefore in the following we will consider the *total* cross-section $\sigma_{\gamma tot}$ for photonuclear interaction.

We have calculated the cross-section σ_μ on the hypothesis that even at high energies (about 10 GeV) $\sigma_{\gamma tot}$ remains constant and equal to $2\text{-}3 \cdot 10^{-28}$ cm/nucleon.

From the observed number of penetrating secondaries and the energy which they must have to cross at least 15 cm of lead between counter-sets B and E we have estimated that the minimum energy ε_m of a photon which can give rise to processes of the kind considered here, is at least 2000 MeV.

In Table III the results of the calculation are compared with the experimental data.

TABLE III. - *Experimental and calculated values of the cross-sections σ_μ on the hypothesis that the total cross-section for photonuclear interaction $\sigma_{\gamma tot}$ remains constant at high energies.*

Depth m w.eq.	σ_μ (exp.) (cm²/nucleon)	σ_μ (calc.) (cm²/nucleon) $\varepsilon_m = 2000$ MeV	
		$\sigma_{\gamma tot} = 2 \cdot 10^{-28}$ (cm²/nucleon)	$\sigma_{\gamma tot} = 3 \cdot 10^{-28}$ (cm²/nucleon)
1	2	3	4
50	$(2.40 \pm 0.4) \cdot 10^{-30}$	$1.40 \cdot 10^{-30}$	$2.11 \cdot 10^{-30}$
200	$(7.50 \pm 1.0) \cdot 10^{-30}$	$4.40 \cdot 10^{-30}$	$6.67 \cdot 10^{-30}$

The experimental values of σ_μ are consistent with the theoretical ones. The agreement is better for $\sigma_{\gamma tot} = 3 \cdot 10^{-28}$ cm²/nucleon, but one has to remember that the experimental values represent upper limits, because we have neglected the contribution of the elctron-photon cascades.

Such a conclusion appears to be weakened by the fact that unfortunately the experimental values of $\sigma_{\gamma tot}$ refer to energies which are of only 300 MeV; i.e. energies much lower than those concerned at depths of 50 and 200 m w.eq..

GEORGE and EVANS (7) observed, with the emulsion technique, that stars, with a number of shower particles varying from 2 to 8 are produced by ionizing primaries. For such events they give a cross-section, valid at the depth of some tens of m w.eq., of $\simeq 1.5 \cdot 10^{-30}$ cm²/nucleon. They also account for the behaviour of the cross-section for production of stars as a function of the underground depth, relying on the results predicted by the theory of Weizsäcker-Williams; they make the hypothesis that the cross-section σ_γ for photo-

(7) E. P. GEORGE and J. EVANS: *Proc. Phys. Soc.*, **63**, 1248 (1950); **64**, 195 (1951).

production of a star remains at a value $\simeq 1.3 \cdot 10^{-28}$ cm²/nucleon even at high energies and with $\varepsilon_m \cong 200$ MeV.

On the other hand BARRET *et al.* ([6]), discussing the results they had obtained with a counter hodoscope at the depth of 1 600 m w.eq., reach the conclusion that there may be a slight increase in the value of $\sigma_{\gamma\text{tot}}$ with the energy of the incident particles. Their discussion is substantially founded on the data they obtained at the depth of 20 and 1 600 m of w.eq. by means of a neutron detector ([6,8]) and on measurements made with the counter hodoscope.

We have also calculated the cross-section σ_μ introducing into the relation (2) for the cross-section for the photoproduction of a π-meson, the theoretical values deduced by ARACKI ([4]) with the pseudoscalar theory with pseudoscalar coupling; according to Araeki's calculations the photonuclear cross-section after having reached a maximum of $\sim 2 \cdot 10^{-28}$ cm²/nucleon at about 200 MeV should slowly diminish as the energy increases. We have taken into account the contribution to the generation of penetrating showers by high energy recoil nucleons which have been produced by photonuclear effects; the calculation leads us to values of σ_μ not very different from those which were obtained on the hypothesis that $\sigma_{\gamma\text{tot}}$ keeps the same value even at high energies. The difference is particularly small for $\varepsilon_m \gtrsim 2\,000$ MeV; the large statistical errors of our experimental results, do not allow us to distinguish between such slight differences.

3. – Conclusions.

1) The number of showers produced in a lead layer of 5 cm and penetrating at least 15 cm Pb, is $(8{,}40 \pm 1) \cdot 10^{-5}$ and $(25.60 \pm 5) \cdot 10^{-5}$ per incident μ-meson, respectively at 50 m w.eq. ($E_0 = 1.14 \cdot 10^{10}$ eV) and 200 m w.eq. ($E_0 = 4.40 \cdot 10^{10}$ eV).

2) Our experimental set-up does not allow to decide what fraction of the observed events should be attributed to normal electron-photon showers. A theoretical calculation shows that, expecially at 50 m w.eq., a good fraction of the observed events could be of this type. The results given in Table I show however that the electron-photon showers cannot explain the appreciable increase in the generation of such events with the average energy of the incident particles. On the other hand the uncertainties by which the calculation on the developement of the shower are affected, are too great to allow a definitive conclusion.

([8]) G. COCCONI and V. COCCONI-TONGIORGI: *Phys. Rev.*, **84**, 29 (1951).

This point could be made clearer by further measurements with an increased layer of lead interposed between the counter sets B and E, as it is indicated by the recent measurements made underground by the research group of Osaka University (9).

3) If we assume that, in spite of the preceeding remark, all the events we have observed are due to nuclear interactions of the incident μ-mesons, we obtain an upper limit for the cross-sections which are still consistent with those predicted assuming an interaction mechanism between the virtual photons which represent the electromagnetic field of the μ-meson and the nucleons. By comparing the results obtained at the two depths considered here, one can conclude that the total photonuclear cross-section does not necessarily undergo an appreciable change by increasing the energy of the incident particles up to a few 10^{10} eV.

Therefore we conclude that the experimental values of the cross-sections and their behaviour as a function of the average energy of the incident particles, can be completely interpreted under the assumption that the only interaction of some importance at energies of the order of a few 10^{10} eV is that due to the electromagnetic field; it seems therefore unnecessary to invoke any new type of interaction of μ-mesons with nucleons.

We would like particularly to thank Prof. E. AMALDI for the interest he has taken in this work and for the numerous discussions we have had with him.

We thank also Mr. CARMELO ZANGHI who has carried out the measurements in the « Castagnola » tunnel.

Our thanks are also due to Mr. ROVELLI of the « Società Romana di Elettricità », who placed at our disposal the tunnel and contributed much material help during the carrying out of the work.

One of us (A.G.) wishes to thank Prof. O. SPECCHIA of the University of Pavia for permission granted to him to remain for several years at the I.N.F.N., Sezione di Roma.

(9) Research group for the cosmic ray underground, Osaka City University (private communication).

RIASSUNTO (*)

La produzione di sciami penetranti a profondità di 50 e 200 m a.e. da parte di mesoni μ di elevata energia è stata studiata con un odoscopio di contatori. I valori della sezione d'urto trovati (in piombo) per eventi di questo tipo si accordano con quelli prevedibili postulando un meccanismo elettromagnetico d'interazione, secondo il quale gli sciami penetranti possono aver inizio dalla fotoproduzione di un mesone π o da un nucleone di rinculo di elevata energia. L'accordo tra la sezione d'urto sperimentale e quella calcolata è ottimo, se si ammette che la sezione d'urto fotonucleare totale, $\sigma_{\gamma tot} = 3 \cdot 10^{-28}$ cm^2/nucleone alle profondità considerate resti invariata anche alle energie in gioco. Non appare pertanto necessario invocare un nuovo tipo di interazione per spiegare i risultati sperimentali.

(*) *Traduzione a cura della Redazione.*

On the L-Capture to K-Capture Ratio in Cd^{109}.

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(ricevuto l'11 Marzo 1954)

Summary. — The L-capture to K-capture ratio in the orbital electron capture occurring in Cd^{109} was measured by means of a proportional counter. This ratio was deduced from the measured intensities of the LX- and KX-radiations, after correction for the counting geometry, the fluorescence yields of Ag, and the internal conversion of 87 keV γ-rays from the atom produced. Thus: $P_L/P_K = 0.32 \pm 0.04$. A test of this result was obtained by measuring the intensities of the KX- and γ-radiations by means of a NaI(Tl) γ-ray spectrometer. In this way: $P_L/P_K \cong 0.29$. Marshak's formulae were used to calculate the transition energy E_0. It was found $E_0 = 67^{+8}_{-3}$ keV.

1. – Introduction.

Of two neighbouring isobars of atomic mass $M(Z)$ ad $M(Z-1)$, the one with atomic number $Z+1$ is unstable against positive β-decay or orbital electron capture provided that:

$$M(Z+1) - M(Z) \geqslant 2m + \mu \,, \tag{1}$$

where m is the mass of the electron, and μ the mass of the neutrino. In many istances the nucleus is unstable against both orbital electron capture and positive β-decay. However the orbital electron capture is an energetically more favourable process as the unstability relation is:

$$M(Z+1) - M(Z) \geqslant \mu + \frac{E_n}{c^2} \,, \tag{2}$$

where E_n is the binding energy (about $^1/_5\, mc^2$ for the heaviest elements) of an electron in the n shell. Usually the electron is supplied by the K shell since

the density of K-electrons at the nucleus is greatest. However L-capture is possible and may become important when the energy of transition, equivalent to the atomic mass difference, is nearly equal to or more than E_K. Obviously the L-capture is found to be the only possibility for the decay of the nucleus, when the nuclear energy change is so low that the K-capture is energetically impossible.

Orbital electron capture can be recognized and measured by observing either Auger electrons or X-rays from the product atom. Clearly these radiations are emitted when the vacancy left in an inner shell by orbital electron capture is filled by one of the outer electrons. Further evidence of the orbital electron decay is shown when the product nucleus is left in an excited level which subsequently emits γ-rays.

The transition energy is carried away by the neutrino and therefore cannot be observed directly. The transition energy can be determined indirectly by measuring the reaction energy Q of the (p, n) reaction which leads from the stable nucleus to the radioactive nucleus. In practice the transition energy is generally unknown.

General formulae for the lifetimes for orbital electron capture were developed by MARSHAK (1) on the basis of the Gamow-Teller interaction in β-decay. By means of results of Marshak's theory it is possible to calculate the transition energy from the L-capture to K-capture ratio.

The detection of L-capture is difficult owing to the softness of both LX-radiations and the Auger electrons emitted. The first measurement of the L-capture (in A^{37}), to evaluate the transition energy, was due to PONTECORVO, KIRKWOOD and HANNA (2).

To measure the ratio between L-capture and K-capture in Cd^{109}, we employed the proportional counter technique. Cd^{109} decays into an isomeric state of Ag^{109} by means of orbital electron capture. The isomeric state has a half-life of 40 s (3) and decays into the ground state through the emission of 87 keV γ-rays.

The decay of Cd^{109} was studied by GUM and POOL (4) over a period of five years and the half life was found to be 470 d. The K-capture was observed for the first time by BRANDT and coworkers (5). No evidence of positron emission was found by DREYFUS, MAJOR and RADVENY (6).

(1) E. R. MARSHAK: *Phys. Rev.*, **61**, 431 (1942).

(2) B. PONTECORVO, D. H. KIRKWOOD and G. C. HANNA: *Phys. Rev.*, **75**, 982 (1949).

(3) E. J. WOLICKI, B. WALDMANN and W. C. MILLER: *Phys. Rev.*, **82**, 486 (1951).

(4) J. R. GUM and M. L. POOL: *Phys. Rev.*, **80**, 315 (1950).

(5) H. BRANDT, P. C. GUGELOT, O. HUBER, H. MEDICUS, P. PREISWERK, P. SCHERRER and R. STEFFEN: *Helv. Phys. Acta*, **19**, 218 (1946).

(6) B. DREIFUS, J. K. MAJOR and P. RADVENY: *Compt. Rend.*, **232**, 617 (1951).

2. – Experimental Apparatus.

The counter employed during these measurements consists of an aluminium cylinder of inner diameter 7 cm and effective length 33 cm, placed in a steel cylinder of 10 cm diameter. The steel tube is provided with two vacuum tight source holders placed at 180° in the middle plane; this arrangement avoids the use of windows. The holders allow the use of a collimated source; they can hold the source on a level with the wall of the counter.

The counter, filled with a 90% Argon an 10% Methane mixture (1.2 atm total pressure), is permanently attached to a purifier which allows the continuous purification of the gas during the measurement ([7]). The purifier consists of a steel tube containing Ca-Mg alloy turnings and is heated electrically to about 300 °C. The alloy has been prepared with calcium and magnesium carefully distilled in vacuum.

The center wire is of high tempered steel, 0.1 mm in diameter; a sufficient thickening of the wire at the two ends, avoids the multiplication process but does not distort much the radial character of the field. The counter is operated by a +3000 V regulated supply. The pulses are taken off the wire and fed through a preamplifier and a mod. 220 linear amplifier ([8]) into a single channel electronic pulse analizer ([9]).

The source was prepared by evaporating on a polythene foil a few drops of the active substance in a carrier free solution of $Cd^{109}Cl$ in diluted HCl. Cd^{109} was produced by irradiation in the Amsterdam's cyclotron.

3. – X-Radiations from Cd^{109}.

The radiations from Cd^{109} are:

a) 87 keV γ-rays.
b) K and L conversion electrons from the 87 keV γ-transition.
c) KX-rays, characteristic of Ag, from the K-capture and from the K internal conversion of 87 keV γ-rays.
d) LX-rays, characteristic of Ag, from the L-capture and from the L internal conversion of 87 keV γ-rays.
e) K and L Auger electrons.

([7]) G. Bertolini, A. Bisi and L. Zappa: *Nuovo Cimento*, **10**, 1424 (1953).
([8]) W. C. Elmore and M. Sands: *Electronic Experimental Techniques* (New York, 1949), p. 170.
([9]) J. E. Francis jr., P. R. Bell and J. C. Gundlack: *Rev. Sci. Instr.*, **22**, 133 (1951).

As our interest was directed to the study of X-radiation, a polythene absorber ≈ 20 mg/cm^2 was placed over the source in order to stop the electrons.

In Fig. 1 the pulse height distribution for KX-radiation and LX-radiation

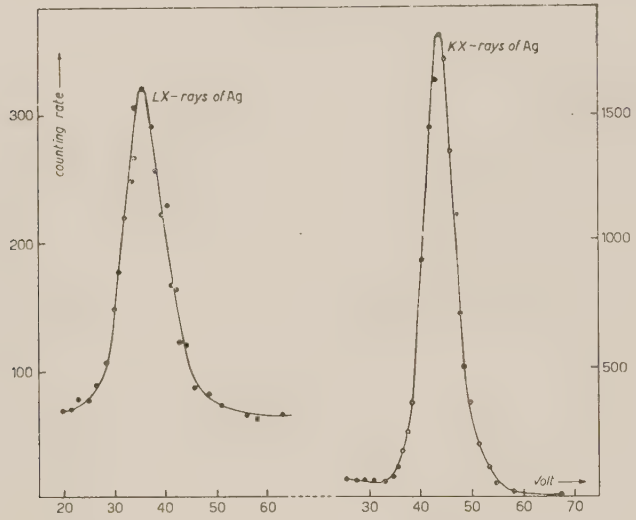

Fig. 1. – Pulse height distribution for K and LX-radiations from Cd^{109} obtained with the proportional counter.

is shown. The peak at 87 keV is absent because of the small efficiency of the argon for detecting γ-radiation in the region over 50 keV.

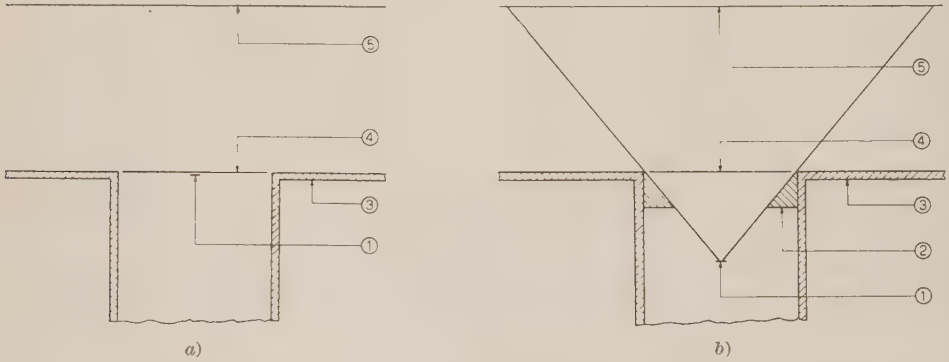

Fig. 2. – Counting geometry (schematic): *a*) Uncollimated source, *b*) Collimated source. 1) Source; 2) Lead plate with conical hole; 3) Counter wall; 4) Al foil; 5) Counter wire.

The LX radiation, because of the small intensity, was studied by placing the source at a level with the wall of the counter (Fig. 2). In spite of this the end effect is absent, in view of the fact that the mean free path for absorption

of the quanta is approximately 0.9 times the counter radius, and because the polythene foil placed over the source absorbs very strongly those quanta which are emitted at large angles with the direction perpendicular to the axis of the counter. Consequently the counting loss coming from the end effects is practically negligible.

The mean free path for absorption of the KX-rays was much greater than the counter dimensions. In order to define exactly the absorbing volume, the source was collimated as shown in Fig. 2. With this counting geometry the emitting foil behaves as a point source of radiation. The ratio f between the counting rate and the disintegration rate can be easily calculated from the dimensions of the counter, from the photoelectric absorption coefficients of the gas and of the absorbers placed over the source. The details of the calculation are shown in the appendix. The results are:

$$(3) \qquad \begin{cases} f(L\mathrm{X}) = 6.74 \cdot 10^{-2}\,, \\ f(K\mathrm{X}) = 1.80 \cdot 10^{-2}\,. \end{cases}$$

Generally the efficiency of a proportional counter for detecting X-radiation, as calculated above, is overestimated because of the wall effect. The photoelectrons from the quanta absorbed in the neighbourhood of the wall, may or may not spend their full energy in the counter gas. Obviously the number of pulses in the peaks of the distribution will be reduced. The counting loss due to the wall effect, depends on the nature and the pressure of the gas. The wall effect for photoelectrons was studied by West, Dawson and Mandleberg ([10]). From the measurements of these workers it is seen that the wall effect, in our experimental apparatus is reduced to practically negligible proportions. Therefore no correction corresponding to this effect was made in the calculation of the efficiency f.

The « measured » intensity I^* of the LX and KX radiation was obtained from the areas under the peaks after subtraction of the background. The pulse height distribution of the background was carefully studied in the range of energy between 1 and 40 keV. From eight determinations we obtain:

$$\frac{I_L^*}{I_K^*} = (1.90 \pm 0.01) \cdot 10^{-1}\,.$$

The « true » values of the ratio I_L/I_K can be therefore obtained from the « measured » ratio and the efficiencies f:

$$4) \qquad \frac{I_L}{I_K} = (5.05 \pm 0{,}03) \cdot 10^{-2}\,.$$

([10]) D. West, J. K. Dawson and C. J. Mandleberg: *Phil. Mag.*, **43**, 875 (1952).

4. – The K shell and L shell fluorescence yields for Ag.

In order to deduce from (4) the ratio between the vacancies produced in the L shell and those in the K shell due to both the orbital electron capture and the γ-internal conversion it is necessary to know the K shell and L shell fluorescence yields.

The K shell fluorescence yields ω_K of Ag has been measured by many workers ([11,12]). We assume:

$$\omega_K = 0.82 \pm 0.01 \ .$$

The fluorescence yields for the subshell L_{I}, L_{II}, L_{III}, are in general different; they have been measured only in the heavy elements ($Z > 72$) ([13]). Therefore, in order to estimate the L fluorescence yields, we used an indirect method due to KINSEY ([14]).

A few words about this method may be useful. The fluorescence yield corresponding to the ionization in an inner shell (K, L...) is defined as the fraction (ω_K, ω_L, ...) of the ionized atoms that reorganize with emission of radiation.

Let Γ and Γ_R be respectively the total width and the partial width for radiative transitions of a state of inner-shell ionization of an atom. Γ and Γ_R are proportional respectively to the total probability and to the partial probability for radiative transitions in which the state of ionization is destroyed. Clearly, then, the fluorescence yield will be: $\omega = \Gamma_R/\Gamma$.

The Γ values can be obtained from the results of the measurements of the total width of X-rays lines and the absorption edges. We used for Ag the measurements made by PARRAT ([15]), (Table I).

The radiative widths Γ_R can be deduced from the relativistic calculations of MASSEY and BURHOP ([16]). Since these authors calculate only the transition probability of some important lines arising from transitions to each L sublevel, KINSEY estimates the contribution of these lines to total width on the basis of Jonsson's intensity measurements. The radiative widths, as calculated

([11]) E. H. S. BURHOP: *The Auger Effect and other Radiationless Transitions* (Cambridge, 1952) p. 45.

([12]) O. HUBER, F. HUMBEL, H. SCHNEIDER and A. DE SHALIT: *Helv. Phys. Acta*, **25**, 3 (1952).

([13]) E. H. S. BURHOP: loc. cit. p. 55.

([14]) B. B. KINSEY: *Canad. Journ. Res.*, **26**A, 404 (1948).

([15]) L. G. PARRAT: *Phys. Rev.*, **54**, 99 (1938).

([16]) H. W. S. MASSEY and E. H. S. BURHOP: *Phil. Cam. Phil. Soc.*, **32**, 451 (1936).

above show a general increase with the fifth power of the atomic number. By using the Kinsey method we obtain for the fluorescence yields ω_{L_I} and $\omega_{L_{III}}$ the values shown in the Table I.

TABLE I.

Subshell	L_I	L_{II}	L_{III}
Total width (volt) . . . . . (PARRAT [15])	5.3	2.2	2.0
Radiation width (volt) . . . . . (MASSEY and BURHOP [16])	0.045	—	1.06
Fluorescence yield	0.0086	0.048	0.028

The fluorescence yield $\omega_{L_{III}}$ is also obtained by extrapolating the results of the measurements of KÜNSTER and ARENDS [17], and STEPHENSON [18]. These measurements show for $\omega_{L_{III}}$ a variation with Z of the type:

$$\omega_{L_{III}} = (1 + \alpha Z^{-5})^{-1}$$

where the value of α is:

$$\alpha = 8.0 \cdot 10^9$$

We obtain, for Ag,

$$\omega_{L_{III}} = 0.028 .$$

In respect of the fluorescence yield $\omega_{L_{II}}$ we estimate that the Γ_R value for Ag ($\Gamma_R \approx 0.06$ V) from the calculations of MASSEY and BURHOP is unreliable. The calculated intensity ratio of the lines arising from transitions to L_{II} level is also unreliable. Therefore we obtained the fluorescence yield $\omega_{L_{II}}$ by extrapolating Kinsey's calculations for elements between Ta (73) and U (92). This value of $\omega_{L_{II}}$ is consistent with a radiative width $\Gamma_R \approx 0.1$ V. The measurements of the total width for the L_{II} level [15, 19-22], show a general

[17] H. KÜNSTER and E. ARENDS: *Ann. der Physik*, **22**, 443 (1935).
[18] R. J. STEPHENSON: *Phys. Rev.*, **51**, 637 (1937).
[19] W. W. BEEMAN and H. FRIEDMAN: *Phys. Rev.*, **56**, 392 (1939).
[20] D. COSTER and K. W. DE LANGEN: *Physica*, **3**, 282 (1936).
[21] J. A. BEARDEN and T. M. SNYDER: *Phys. Rev.*, **59**, 162 (1941).
[22] F. K. RITCHMEYER, S. W. BARNES and G. E. RAMBERG: *Phys. Rev.*, **46**, 843 (1934).

increase of level widths with the atomic number, approximately of the type:

$$\Gamma = A + BZ^5 ,$$

where the two terms represent respectively the contribution from radiationless and radiative transitions. The term BZ^5, for Ag, is ≈ 0.1 V.

5. – The L-capture to K-capture ratio.

Let us indicate with N_L and N_K the number of vacancies, respectively produced in the L-shell and K-shell by the orbital electron capture and by the γ-internal conversion. Then:

$$\frac{I_L}{I_K} = \left[(\omega_L)_K + (\omega_L)_L \frac{N_L}{N_K} \right] \frac{1}{\omega_K} ; \tag{5}$$

$(\omega_L)_K$ is the fluorescence yield for transitions to the L levels, following the K-capture and the K-internal conversion of the γ-rays;
$(\omega_L)_L$ is the fluorescence yield for transitions to the L levels following the L-capture and the L-internal conversion of the γ-rays;
$(\omega_L)_K$ and $(\omega_L)_L$ can be calculated from the fluorescence yields for the subshells L_{I}, L_{II}, L_{III}, provided the relative excitations (u and v) of the L levels are known. Hence:

$$\begin{cases} (\omega_L)_K = u_1 \omega_{L_{\mathrm{I}}} + u_2 \omega_{L_{\mathrm{II}}} + u_3 \omega_{L_{\mathrm{III}}} , \\ (\omega_L)_L = v_1 \omega_{L_{\mathrm{I}}} + v_2 \omega_{L_{\mathrm{II}}} + v_3 \omega_{L_{\mathrm{III}}} , \end{cases} \tag{6}$$

where [23,24]:

$$u_1 = 0 , \qquad u_2 = \frac{1}{3} , \qquad u_3 = \frac{2}{3} ,$$

$$v_1 = \frac{100}{111} , \qquad v_2 = \frac{10}{111} , \qquad v_3 = \frac{1}{111} .$$

From (5) and (6) we obtain:

$$\frac{N_L}{N_K} = 0.56 \pm 0.03 . \tag{7}$$

([23]) E. H. S. BURHOP: loc. cit., p. 51.
([24]) M. M. MILLER and R. G. WILKINSON: *Phys. Rev.*, **83**, 1050 (1951).

The P_L/P_K ratio between the probabilities for the L-capture and the K-capture is then calculable from (7) and from the internal conversion coefficients of 87 keV γ-rays listed in the following table:

TABLE II.

$\alpha_K = 8.6 \pm 1$	AVIGNON [25]
$\alpha_K/\alpha_L = 1.01$	BRANDT *et al.* [5]
$\alpha_K/(\alpha_L + \alpha_M) = 6.1$	BRANDT *et al.* [5]

Thus:

$$\frac{P_L}{P_K} = 0.32 \pm 0.04 \,. \tag{8}$$

The error in (8) was estimated from the uncertainty of the values of the fluorescence yields and of the internal conversion coefficients.

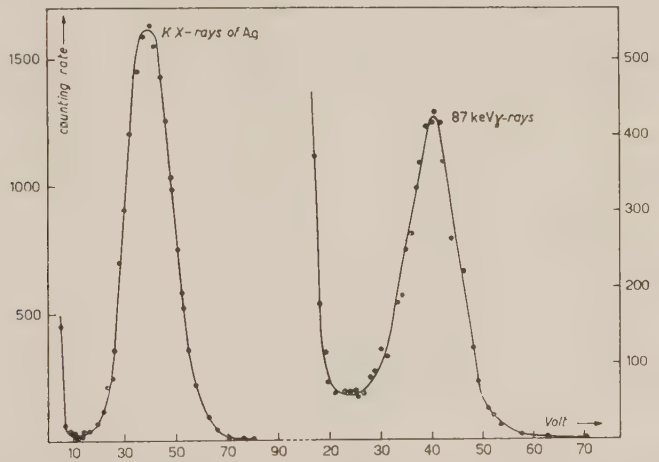

Fig. 3. – Pulse height distribution of KX- and γ-radiations of Cd^{109} obtained with the scintillation spectrometer.

The decay scheme of Cd^{109} allows, as a test of this result, a further determination of the P_L/P_K ratio, indipendent from the L-shell fluorescence yields of Ag. This determination was made by measuring the relative intensities of the γ and KX radiations by means of a single crystal spectrometer (NaI-Tl activated). The crystal (cylindrical, 1″ diameter, 1″ thick) was fitted to the photocathode of a phototube RCA 5819. In order to define exactly the counting geometry, the source was collimated. Fig. 3 shows the pulse height distri-

[25] P. AVIGNON: *Journ. de Phys. et Rad.*, **14**, 636 (1953) .

bution of KX and γ-radiations. The intensities were obtained from the areas under the peaks, and the ratio between the counting rate and the disintegration rate was evaluated from the photo-electric absorption coefficients of the absorbers placed over the source. The efficiency of detection of the crystal for the observed radiations is approximately 100 percent. The ratio of the intensities of the two lines was found to be: $I_K/I_{\gamma} \cong 18$ and consequently:

$$\frac{P_L}{P_K} \cong 0.29 \,. \tag{9}$$

Because the estimation of the error in (9) is very rough, in the following section we shall use the value (8).

Another test of our result is given by the result of DER MATEOSIAN's [26] measurements printed at the end of our measurements. DER MATEOSIAN used a source of Cd^{109} dispersed throughout a NaI(Tl) crystal, mounted on a photomultiplier tube. The ratio between the number of K-captures and the total number of disintegrations is measured; by difference the number of L-captures is obtained. Then: $P_L/P_K = 0.28 \pm 0.03$.

Clearly the agreement of the two measurements is quite satisfactory.

6. – Conclusion.

From the P_L/P_K ratio, as shown in section **1**, it is possible to obtain the transition energy for orbital electron capture. When Marshak's general formulae for allowed transitions are used, we have:

$$\frac{P_L}{P_K} = \left(\frac{\psi_{L_1}}{\psi_K}\right)^2 \left[\frac{W_0 + W_L}{W_0 + W_K}\right]^2 , \tag{10}$$

where $(\psi_{L_1}/\psi_K)^2$ is the ratio of L- and K-shell electron densities at the nuclear radius. W_K and W_L are the K- and L-shell energies ($W_K = 1 - {}^1/_2\,\alpha^2 Z_{\text{eff}}^2$; $W_L = 1 - {}^1/_8\,\alpha^2 Z_{\text{eff}}^2$; $\alpha = 1/\text{«}\,137\,\text{»}$; $Z_{\text{eff}} = Z - 0.3$); W_0 is the nuclear energy change in units of mc^2.

The $(\psi_{L_1}/\psi_K)^2$ ratio has been calculated by ROSE and JACKSON as a function of Z [27].

With $P_L/P_K = 0.32$, we obtain for the transition energy E_0:

$$E_0 = 67^{+8}_{-3}\ \text{keV}.$$

[26] E. DER MATEOSIAN: *Phys. Rev.*, **92**, 938 (1953).

[27] M. E. ROSE and J. L. JACKSON: *Phys. Rev.*, **76**, 1540 (1949).

The log ft from our value of nuclear energy change, according to Feenberg and Trigg's formula ([28]), is:

$$\log ft = 5.0 .$$

This value is consistent with an allowed transition.

We are greatly indebted to prof. G. Bolla, director of the Institute, for his stimulating interest in the present work.

Appendix

The ratio between counting rate and disintegration rate has been calculated as follows:

a) *Uncollimated source.* – Let us introduce a polar coordinate system, with its origin in the source, its polar axis along the radius R, and its plane $\varphi = 0$ perpendicular to the axis of the counter. The equation of the cylinder is then:

$$\varrho^2 \sin^2\theta \cos^2\varphi + \varrho^2 \cos^2\theta - 2R\varrho \cos\theta = 0$$

and the track length of quanta in crossing the counter is:

$$\varrho = \frac{2R \cos\theta}{\sin^2\theta \cos^2\varphi + \cos^2\theta} .$$

Let h_1 and μ_1 be respectively the thickness and the photo-electric absorption coefficient of the absorber placed on the source, μ_2 the photo-electric absorption coefficient of the counter gas. Clearly the fraction of quanta absorbed in the counter gas will be:

$$f = \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} \frac{\sin\theta \, d\theta \, d\varphi}{2\pi} \exp[-\mu_1 h_1 / \cos\theta] \cdot (1 - \exp[-\mu_2 \varrho]) .$$

b) *Collimated source.* – Let us again introduce a polar coordinate system, with its origin in the source, its polar axis along the radius, and its plane $\varphi = 0$ perpendicular to the axis of the counter. The equation of the cylinder is now:

$$\varrho^2 \sin^2\theta \cos^2\varphi + \varrho^2 \cos^2\theta - 2D\varrho \cos\theta + D^2 - R^2 = 0 ,$$

$D+R$ is the distance between source and wire of the counter.

([28]) E. Feenberg and G. Trigg: *Rev. Mod. Phys.*, **22**, 399 (1950).

Thus the track length of quanta in crossing the counter will be:

$$\varrho^* = \frac{2}{\cos^2\varphi + \sin^2\varphi \cos^2\theta} \{R^2(\cos^2\varphi + \cos^2\theta \sin^2\varphi) - D^2 \cos^2\varphi(1 - \cos^2\theta)\} .$$

We have then:

$$f = \int_{\theta=0}^{\theta^*} \int_{\varphi=0}^{2\pi} \frac{\sin\theta \, d\theta \, d\varphi}{2\pi} \exp[-\mu_1 h_1 / \cos\theta] \cdot (1 - \exp[-\mu_2 \varrho^*]) ,$$

where h_1, μ_1, μ_2 have the same definition as in *a*) and θ^* is the maximum value of θ.

The integrals were calculated using Gauss quadrature formulae.

RIASSUNTO

Si riferisce sul decadimento del Cd^{109} per cattura elettronica orbitale. Il rapporto P_L/P_K tra la probabilità di disintegrazione per cattura L e quella per cattura K viene determinato misurando con un contatore proporzionale l'intensità delle radiazioni LX e KX. Si ottiene: $P_L/P_K = 0{,}32 \pm 0{,}04$. Una conferma di questo risultato viene ottenuta misurando le intensità delle radiazioni KX e γ con uno spettrometro a scintillazione ad un cristallo. Questo metodo fornisce il valore: $P_L/P_K \simeq 0{,}29$. Utilizzando la teoria di Marshak sul decadimento per cattura elettronica, si ha per l'energia della transizione: $E_0 = 67^{+8}_{-3}$ keV.

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Non-Mesonic Decay of a Bound V_1^0-Particle. (*)

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(ricevuto il 15 Marzo 1954)

Summary. — A charged nuclear fragment, ejected from a cosmic ray star, stopped in the emulsion and subsequently disintegrated into two doubly charged particles. Both of the particles from the disintegration stopped in the emulsion. From the energetics of the disintegration and the characteristics of the tracks, the fragment was found to have been $(Be^7+V_1^0)_{bound}$, or $(Be^8+V_1^0)_{bound}$, with a Q of 170 ± 3 MeV or 176 ± 3 MeV respectively. The nuclear binding of the V_1^0 was 4 ± 5 MeV or -1 ± 5 MeV respectively.

1. – Introduction.

Several events have been found in photographic emulsion by various investigators [1-6] where a charged nuclear fragment appears to have stopped

(*) Supported in part by the Graduate Committee from funds supplied by the Wisconsin Alumni Research Foundation.

(+) National Science Foundation Predoctoral Fellow. Work performed at the University of Wisconsin.

(1) M. Danysz and J. Pniewski: *Phil. Mag.*, **44**, 348 (1953).

(2) D. A. Tidman, G. Davis, A. J. Hertz and R. M. Tennent: *Phil. Mag.*, **44**, 350 (1953).

(3) J. Crussard and D. Morellet: *Compt. Rend.*, **236**, 64 (1953).

(4) G. Lovera, L. Barbanti Silva, C. Bonacini, C. De Pietri, R. Perilli Fedeli and A. Roveri: *Nuovo Cimento*, **10**, 986 (1953).

(5) P. S. Freier, G. W. Anderson and J. E. Naugle: *Phys. Rev.* (to be published).

(6) R. D. Hill, E. O. Salant and M. Widgoff, L. S. Osborne, A. Pevsner and D. M. Ritson, J. Crussard and W. D. Walker: *Bull. Am. Phys. Soc.*, **29**, 60A (1954).

and subsequently disintegrated. These events are interpreted as the decay of a V_1^0 which had been bound in the nuclear fragment (7).

2. – Description.

A disintegration of a nuclear fragment was found among 2 300 high energy cosmic ray stars. The event occurred in a 1000 micron pellicle which was exposed in the stratosphere by a « skyhook » balloon flight. Fig. 1 is a photograph of the event. The initial star *A* has 22 prongs. There are 5 minimum ionizing tracks from star *A*. The fragment *F* appears to stop at point *B*. The secondary star *B* has two prongs.

3. – Analysis.

Track *F* is 55 μ long. Although the track is short, the multiple scattering is clearly inconsistent with that of a heavy meson. It is similar to the scattering near the end of Be tracks. A comparison (8) of the thin down along track *F* with that of beryllium and carbon shows that the charge of fragment *F* was less than 6 and greater than 1. The general appearance of the track indicates that the nuclear fragment stopped at point *B*. Star *B* has only two prongs, with no visible recoil track. Both prongs end in the emulsion.

The lateral expansion factors of the pellicle were determined from the dimensions of the emulsion before and after processing. These factors were checked by measuring π- μ-meson decay ranges in the same pellicle and were found to be consistent with the mean range in normal plates after applying the expansion factor. The ranges of tracks 1 and 2, corrected for lateral expansion, are 748 ± 4 and 1011 ± 5 μ respectively. The angle between tracks is $145.8 \pm 0.4°$.

Delta ray counts on tracks 1 and 2 prove that each was caused by a particle with $Z = 2$. The mass of a particle of known Z can be determined from the intercept of the δ-ray number vs. residual range curve. With 15 to 1 odds, the mass of particle 1 is greater than 3; most likely it is an α-particle. It is equally probable that particle 2 has a mass of 3 or of 4. The mass resolution of this method was obtained by comparing intercepts from several star prongs of $Z = 2$ in the same plate. The short lifetime of He^5 and the absence of electrons from the end of tracks 1 or 2 eliminate the possibility that the tracks were due to He^5 or He^6.

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(7) W. Cheston and H. Primakoff: *Phys. Rev.*, **92**, 1537 (1953).
(8) C. N. Chou, W. F. Fry and J. J. Lord: *Phys. Rev.*, **87**, 671 (1952).

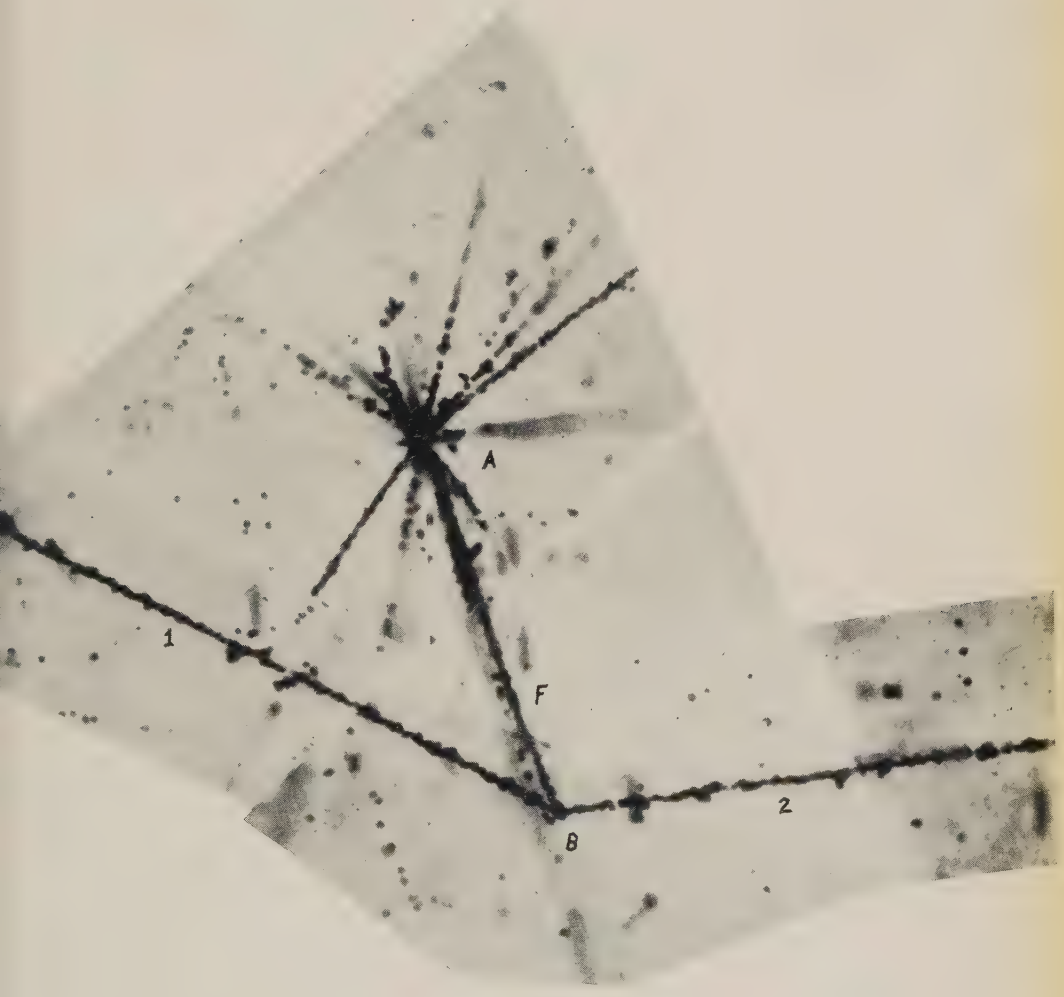

Fig. 1.

The vector sum of the momentum of particles 1 and 2 is 369 ± 3 MeV/c if both particles are assumed to be α-particles; it is 368 ± 3 MeV/c if particle 1 is assumed to be an α and particle 2 to be He^3. Since the charge of fragment F is estimated to be less than 6 from thin down, the residual charge from star B is then 2 or less. With 369 MeV/c residual momentum, it is extremely improbable that a particle with charge 1 would not be seen; therefore the charge of the fragment is assumed to be 4.

4. – Interpretation.

Since the charged particles stopped in the emulsion it is possible to determine their energies quite accurately, which permits a detailed analysis of the event. The residual momentum of star B could have been carried off by one or more neutral particles. The event may be due to one of the following reactions,

$$(Be^7 + V_1^0)_{\text{bound}} \to He^3 + He^4 + n\,,$$

$$(Be^8 + V_1^0)_{\text{bound}} \to He^4 + He^4 + n\,.$$

Although Be^8 is unstable by 95 keV, it is quite possible that $(Be^7 + V_1^0)_{\text{bound}}$ is stable.

The kinetic energy of the reaction products was found to be 170 ± 3 MeV and 176 ± 3 MeV respectively. The energy of the neutron was calculated from the residual momentum of the two charged particles. The expected excitation energy for a bound V_1^0 is $139 + 37 - 1.3 = 175 \pm 2$ MeV [9] if the π-meson is absorbed in the decay.

The similar values of the excitation energy and the kinetic energy of reaction products strongly support the hypothesis that only one neutron was involved in the disintegration. The difference between the excitation energy (175 ± 2 MeV) and the kinetic energy will depend on the binding of the V_1^0 and the manner of disintegration of the fragment.

Two additional reactions are possible but improbable from phase space considerations,

$$(Be^8 + V_1^0)_{\text{bound}} \to He^3 + He^4 + 2n\,,$$

$$(Be^9 + V_1^0)_{\text{bound}} \to He^4 + He^4 + 2n\,.$$

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[9] F. M. Smith, W. Birnbaum and W. H. Barkas: *Phys. Rev.*, **91**, 765 (1953); H. S. Bridge, C. Peyrou, B. Rossi and R. Safford: *Phys. Rev.*, **91**, 362 (1953). The expected excitation is the π-meson rest energy plus the Q of the V_1^0 minus the neutron-proton mass difference.

If the excitation is assumed to be 175 MeV the available phase space is limited by the small maximum energy of one of the neutrons.

5. – Conclusions.

The close agreement of the energetics with the expected excitation show that the star was caused by the non-mesonic decay of a bound V_1^0-particle.

If the nuclear fragment was $(Be^8 + V_1^0)_{bound}$ the binding energy of the V_1^0 was -1 ± 5 MeV. A neutron in Be^9 is bound with 1.7 MeV.

If the nuclear fragment was $(Be^7 + V_1^0)_{bound}$ the binding energy of the V_1^0 was 4 ± 5 MeV. A neutron in Be^8 is bound with 19 MeV.

In either case the general conclusion can be drawn that a V_1^0 particle is not tightly bound in a Be nucleus.

6. – Acknowledgments.

The plates were flown by Major DAVID G. SIMONS USAF. His continued interest and cooperation are greatly appreciated.

RIASSUNTO (*)

Un frammento nucleare carico, emesso da una stella generata da raggi cosmici, si è fermato nell'emulsione disintegrandosi, successivamente, in due particelle dotate di carica doppia. Ambedue le particelle originate nella disintegrazione si sono arrestate nell'emulsione. Dal bilancio energetico della disintegrazione e dalle caratteristiche delle tracce, si è dedotto che il frammento deve essere stato in origine $(Be^7 + V_1^0)_{leg}$ oppure $(Be^8 + V_1^0)_{leg}$ con una Q di 170 ± 3 MeV o di 176 ± 3 MeV, rispettivamente. Il legame nucleare della V_1^0 è di 4 ± 5 MeV, oppure -1 ± 5 MeV, rispettivamente.

(*) *Traduzione a cura della Redazione.*

LETTERE ALLA REDAZIONE

(La responsabilità scientifica degli scritti inseriti in questa rubrica è completamente lasciata dalla Direzione del periodico ai singoli autori)

Proposal of a Synchrotron with a Double Vacuum Chamber.

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(ricevuto il 20 Marzo 1954)

1. – The diameter of the beam of particles which is accelerated in a synchrotron is a function, among other variables, of the energy of these particles In fact the theory indicates that:

— The amplitude of the betatron oscillations varies as $H^{-\frac{1}{2}}$: (H is the flux density) [1].

— The amplitude of the synchrotron oscillations is damped in time at a rate proportional to $E^{-\frac{1}{2}}$: (E, total energy) [1].

— The betatron oscillations which are due to the atomic collisions with the residual gas are important only at low energies.

— The effect of eddy currents, inhomogeneities, etc., is mainly important at low energies.

On the other side, the reduction in diameter of the beam as its energy increases is an observed experimental fact, For instance in Brookhaven the beam has a final diameter of only 1 inch [2,3]. This diameter does not change appreciably during the spiralisation of the beam toward the target [2], notwithstanding the beam makes 4000 turns during that time.

2. – On the basis of these considerations we suggest a synchrotron with two different gaps (or chambers) and two different acceleration stages.

The cross-section of such a machine is illustrated in Fig. 1. The operation goes as follows:

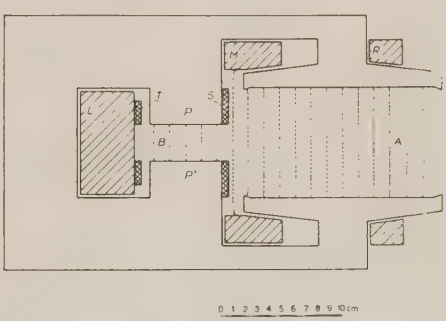

Fig. 1. – Cross-section (orthogonal to the beam) of the synchrotron with two separate chambers. The lines in chambers A and B indicate the field we obtained (n. 4) with an electric model.

a) The particles are injected through a deflector into the chamber A and accelerated up to a moderate energy: for

(1) J. H. FREMLIN and J. S. GOODEN: *Rep. on Progr. in Phys.*, **13**, 295 (1950).

(2) M. HILDRED BLEWETT: *Rev. Scient. Inst.*, **24**, 725 (1953).

(3) C. E. SWARTZ: *Rev. Scient. Inst.*, **24**, 851 (1953).

instance particles electrons, injected at 2-3 MeV, are accelerated in A up to 60 MeV; with a radius of the synchrotron of 3.3 meters the flux density in A reaches about 600 gauss. The dimensions of our drawings refer to such an example;

b) As soon as the particles reached this energy (for instance, just 60 MeV) the beam shall slide from chamber A to chamber B: such a sliding will be possible if the flux density H in the median plan will follow the law:

$$H = H_0 \left(\frac{r_0}{r}\right)^n \qquad 0 < n < 1 ,$$

in both chambers A and B at the same time. The sliding may be obtained for instance by interrupting the Radio Frequency in the resonant cavity as the flux density increases (we suppose that the chamber A is on the outside) or by controlling the frequency and amplitude modulation of the Radio Frequency. On the basis of what we reported in n. 1 the sliding should not spoil the beam. As soon as the beam reaches the chamber B, a new resonant cavity or the same one with a different frequency will keep the beam in B, and the particles will be accelerated up to the maximum flux density (for instance, following our example, 10000 gauss, corresponding to 1000 MeV). During this time the increase of the field in A may be stopped or slowed down without danger for the beam, since we do not need the chamber A any more.

3. – In a conventional synchrotron the dimensions of the gap (or at least its vertical dimensions) remain the same during all the acceleration process although we need large dimensions during the first part of the acceleration only (n. 1). With our proposal we hope to reduce the large amounts of iron and excitation energy required to obtain the maximum flux density in a gap of the dimensions of chamber A (Fig. 1); in fact high flux density will be reached in chamber B only, and this chamber may be much smaller.

TABLE I.

Comparison between a conventional weak-focusing synchrotron (column I*) which has a gap of the dimensions of our chamber A (fig.* 1, 2*) and the synchrotron we suggest here (column* II*). The numbers in column* I *and* II *refer to the example of an electron-synchrotron that we give in the text.*

	I	II
Maximum energy (MeV)	1000	1000
Average radius (cm)	330	330
Weight of iron (kg)	$7 \cdot 10^4$	$\sim 1.6 \cdot 10^4$
Weight of copper (kg)	$1\text{-}2 \cdot 10^4$	$\sim 5 \cdot 10^3$
Total energy stored in the gap (Joule) . . .	$\sim 2.4 \cdot 10^5$	$\sim 5 \cdot 10^4$

In Table I we compare one weak focusing synchrotron which has a gap of the dimensions of chamber A with the two-chamber synchrotron we propose here; as we see, the saving in materials and cost may be of a factor three or more. This factor does not depend critically on the value we choose for the ratio between the dimensions of the chamber A and the radius of the synchrotron.

In Fig. 2 we sketched the cross-sections of a conventional synchrotron (Fig. 2 *a*)) and of that which we propose

here, giving in Fig. 2 *b*) and 2 *c*) two of the possible alternatives. Since the sections are in the same scale and for the same supposed radius, the possible advantages in cost and dimensions are quite evident.

We cannot exclude that the hypothetical machine we are considering here theory, and may be solved in practice by putting the right ampere-turns difference between the equipotential surfaces (magnetic poles) defining the chambers A and B. A possible schema of the excitation may be for instance the following, if the dimensions of our example in Fig. 1 are used:

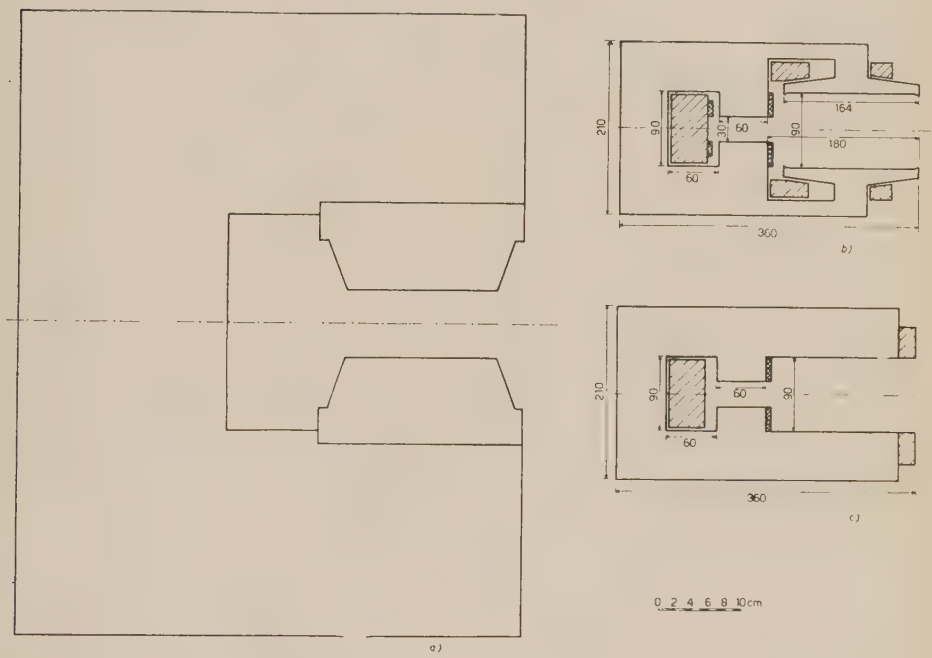

Fig. 2. – Comparison (in the same scale) between the dimensions of a conventional synchrotron (Fig. 2*a*) with a gap of the dimensions of our chamber A, and the synchrotron with two chambers (Fig. 2 *b*). In Fig. 2 *c* another solution for the shape of the two-chamber synchrotron is indicated.

could compete with the strong focusing synchrotron in the 10 GeV region (4).

4. – Among the fundamental questions to be solved we recall the following:

a) *The shape of the magnetic field.* The magnetic field has to be almost vertical and uniform ($0 < n < 1$) in both chambers A and B. This is possible in

Current-turns in the coil $R = \quad 9\ f(t)$
Current-turns in the coil $S = -6\ f(t)$
Current-turns in the coil $T = -3\ f(t)$

$f(t)$ is a convenient function of time. As the beam enters in B the coils L and M will bring the flux density in B to the maximum value (for instance 10 000 gauss). Of course this is a schematic division, and the same coils may be used for different purposes in different times.

To have a first confirmation on the possibility of reaching at the same time the almost uniform magnetic field we

(4) We wish to remember a proposal of OLIPHANT of combining an air core proton synchrotron with a concentric synchro-cyclotron. M. L. OLIPHANT: *Nature*, **165**, 466 (1950).

need in the chambers A and B, we made a map of the field with a conjugate model similar to the electrolitic tank method. This method has been developed by Dr. F. AMMAN and will be discussed elsewhere (5). The results of this preliminary approach were even better than we could hope; in fact when we used parallel poles for A and B ($n=0$) the flux density resulted to be uniform in both chambers A and B, including the transition region, in the limits of a few percent. The lines of flux are indicated in Fig. 1.

(5) This method mainly consists in drawing the equipotentials in a field of currents traversing a thin aluminium foil whose boundaries are the boundaries of the chambers A, B.

b) Vacuum chamber. The doughnut will result of an unusual and difficult shape. The possibility of using stainless steel could solve the problem. The effect of the eddy currents is reduced as the flux density increases, and could be already small when the beam enters the chamber B; therefore the doughnut could consist of stainless steel at least in B.

c) Radio Frequency acceleration. The klystron-type cavities in the straight sections may probably be cut in a way which allows the passage of the doughnut.

5. – Application of similar concepts to a synchrotron working with alternate gradient magnets (strong focusing) is under consideration.

Einzeitige Verfahren der Feldphysik.

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(ricevuto il 5 Aprile 1954)

Die mathematischen Methoden der Feldphysik lehnen sich heute noch weitgehend an die wohlausgebaute Theorie der wechselwirkungsfreien quantisierten Felder an. Das gilt auch für die gerade in letzter Zeit weiterentwickelten einzeitigen Methoden, in denen ein Zustand Φ durch einzeitige oder zeitunabhängige Funktionen beschrieben wird, die mehr oder weniger zweckmäßige Verallgemeingerungen der wohlbekannten Wellenfunktionen des wechselwirkungsfreien Falls sind. Zweck der vorliegenden Note ist, am Beispiel der Wechselwirkung von Nukleonen mit Mesonen einen Überblick der zur Zeit bekannten einzeitigen Methoden zu geben.

Wohl die einfachste Möglichkeit, die wechselwirkungsfreien Wellenfunktionen zu verallgemeinern, ist, einen Zustand Φ des Wechselwirkungsproblems durch die sogenannten Wahrscheinlichkeitsamplituden

$$(1) \qquad \chi(\boldsymbol{k}_1 \dots | \boldsymbol{p}_1 \dots | \boldsymbol{q}_1 \dots) = (\Omega_0, \psi_0^+(\boldsymbol{p}_1) \dots A_0^+(\boldsymbol{k}_1) \dots \bar{\psi}_0^+(\boldsymbol{q}_1) \dots \Phi) \quad (1)$$

darzustellen ([2-6]). (Ω_0 ist das wechselwirkungsfreie Vakuum, $\psi_0^+(p), \dots$, sind die Vernichtungsoperatoren der freien Felder). Das läuft bekanntlich auf eine Entwicklung von Φ nach Energie-Impulseigenvektoren des wechselwirkungsfreien Problems hinaus. Trotz des damit verbundenen Vorzugs begrifflicher Klarheit sind die Wahrscheinlichkeitsamplituden, da sie auf das wechselwirkungsfreie Vakuum Bezug nehmen, beim gegenwärtigen Stand der Feldphysik unbrauchbar. Denn gerade die Beziehung auf Ω_0 bringt zusätzliche Divergenzschwierigkeiten (Divergenzen vom Stueckelberg'schen Typ), die sich im Rahmen des Renormierungsprogramms nicht beseitigen lassen ([7,8]).

Divergenzen dieser Art werden ganz von selbst vermieden, wenn die verallgemeinerten Wellenfunktionen auf das *echte* Vakuum Ω der Gesamtenergie bezogen

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(1) Permutationsfaktoren werden in die Definition der Wahrscheinlichkeitsamplituden und Wellenfunktionen nicht aufgenommen.

(2) V. Fock: *Zeits. f. Phys.*, **75**, 622 (1932).

(3) I. Tamm: *Journ. Phys. (U.S.S.R.)*, **9**, 449 (1950).

(4) S. M. Dancoff: *Phys. Rev.*, **78**, 382 (1950).

(5) M. Lévy: *Phys. Rev.*, **88**, 72 (1952).

(6) M. Cini: *Nuovo Cimento*, **10**, 526 u. 614 (1953).

(7) H. Lehmann: *Zeits. f. Naturf.*, **8a**, 579 (1953).

(8) W. Zimmermann: *Suppl. al Nuovo Cimento* (im Erscheinen).

sind. Das geschieht z.B., wenn die wechselwirkungsfreie Zwei-Nukleonenwellenfunktion

$$\varphi_t^0(|\boldsymbol{x}_1\boldsymbol{x}_2|) = (\Omega_0, \psi_0^+(x_1)\psi_0^+(x_2)\Phi_0)_{x_i^0=t} = (\Omega_0, \psi_0(x_1)\psi_0(x_2)\Phi_0)_{x_i^0=t}$$

durch

$$\varphi_t(|\boldsymbol{x}_1\boldsymbol{x}_2|) = (\Omega, \psi(x_1)\psi(x_2)\Phi)_{x_i^0=t} \tag{2}$$

auf den Wechselwirkungsfall übertragen wird. [$\psi(x)$ Feldoperator der Heisenbergdarstellung]. Um einzeitige (bzw. zeitunabhängige) Wellenfunktionen dieser Eigenschaft für beliebige Koordinatenzahl einzuführen, hat man zwei Möglichkeiten, die in trivialer Weise zusammenhängen:

1.) Man bildet die Funktionen

$$(\Omega, \psi_0^-(x_1') \dots A_0^-(y_1') \dots \bar{\psi}_0^-(z_1') \dots \psi_0^+(x_1) \dots A_0^+(y_1) \dots \bar{\psi}_0^+(z_1)\Phi)_{x_i'^0=y_i'^0=z_i'^0=x_i^0=y_i^0=z_i^0=0} \tag{3}$$

($\psi_0^-; \dots, \psi_0^+; \dots$ sind Erzeugungs- und Vernichtungsoperatoren der wechselwirkungsfreien Felder zur Zeit $t=0$). Aus den Energieeigenwertgleichungen von Ω und Φ, sowie aus den Vertauschungsrelationen der wechselwirkungsfreien Operatoren folgt ein unendliches Gleichungssystem (I) der Funktionen (3).

2.) Aus den invarianten «mehrzeitigen Wellenfunktionen» ([8,11-13]) werden durch Grenzübergang zu gleichen Zeiten «einzeitige Wellenfunktionen» definiert:

$$\varphi_t(\boldsymbol{y}_1 \dots |\boldsymbol{x}_1 \dots |\boldsymbol{z}_1 \dots) = \lim_{x_i^0, y_j^0, z_k^0 \to t} \varphi(y_1 \dots |x_1 \dots |z_1 \dots)\,. \tag{4}$$

Ein unendliches Differentialgleichungssystem (II) ([8,14]) der einzeitigen Wellenfunktionen wird aus dem invarianten Differentialgleichungssystem ([8,11-13]) der mehrzeitigen Wellenfunktionen durch Gleichsetzen der Zeiten gewonnen. Ein weiteres unendliches Integralgleichungssystem (III) der einzeitigen Wellenfunktionen wird durch Multiplikation der Yang-Feldmangleichungen und Umrechnung einzeitiger Operatorprodukte auf Wellenfunktionen erhalten ([10,15]). Die Systeme (I), (II) und (III) sind äquivalent, bis auf den Zusatz, daß (III) noch die Randbedingungen für $t=-\infty$ bzw. $+\infty$ berücksichtigt.

Zur Lösung dieser unendlichen Gleichungssysteme hat man folgendes Näherungsverfahren (Neue Tamm-Dancoffmethode) ([9,14]). Die einzeitigen Wellenfunktionen werden von einer gewissen Koordinatenzahl ab gleich Null gesetzt und das so entstandene *endliche* Gleichungssystem an Stelle des unendlichen Systems weiter behandelt. So vernachläßigt man z.B. in der untersten Näherung des Zwei-Nukleonenproblems alle Wellenfunktionen mit mehr als zwei Nukleonen- und einer Mesonen-

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([9]) F. J. Dyson: *Phys. Rev.*, **91**, 1543 (1953).

([10]) Funktionale Fassung der Gleichungssysteme (I) und (III) in K. Symanzik: *Dissertation Göttingen.*

([11]) K. Nishijima: *Progr. Theor. Phys.*, **10**, 549 (1953).

([12]) E. Freese: *Zeits. f. Naturf.*, **8a**, 776 (1953).

([13]) P. T. Matthews und A. Salam: *Proc. Roy. Soc.*, A. **221**, 128 (1954).

([14]) W. Heisenberg: *Nachr. Akad. Wiss., Gött., Mat. Phys. Kl.*, p. 111 (1953).

([15]) W. Zimmermann: *Nuovo Cimento* (im Erscheinen).

koordinaten, und hat dann statt des unendlichen Systems zwei bzw. drei gekoppelte Gleichungen.

Aus dem System (III) und der Randbedingung, daß Φ die Streuung zweier Nukleonen der Wellenfunktion $\varphi_t^0(|\boldsymbol{x}_1\boldsymbol{x}_2|)$ beschreibt, erhält man durch Zurückführen aller Wellenfunktionen auf $\varphi_t(|\boldsymbol{x}_1\boldsymbol{x}_2|)$ die Integralgleichung

$$(5)\qquad \varphi_t(|\boldsymbol{x}_1\boldsymbol{x}_2|) = Z_1\varphi_t^0(|\boldsymbol{x}_1\boldsymbol{x}_2) + \int_{-\infty}^{t} \mathrm{d}t' \int \mathrm{d}\boldsymbol{x}_1'\mathrm{d}\boldsymbol{x}_2' \, K(\boldsymbol{x}_1\boldsymbol{x}_2 t;\, \boldsymbol{x}_1'\boldsymbol{x}_2' t')\varphi_{t'}(|\boldsymbol{x}_1'\boldsymbol{x}_2'|)\,,$$

in der K eine Entwicklung nach Potenzen der Kopplungskonstanten g_0 ist, deren Glieder sich durch Dyson'sche Doppelgraphen veranschaulichen lassen ([15]). Gebundene Zustände werden durch die zugehörige homogene Integralgleichung beschrieben. Während in der g_0^2-Näherung von K Gl. (5) der erwähnten untersten Näherung der neuen Tamm-Dancoffmethode äquivalent ist, besteht in den höheren Näherungen kein so einfacher Zusammenhang. Schon die nächste Näherung der neuen Tamm-Dancoffmethode, in der auch Wellenfunktionen mit zwei Mesonen- und zwei Nukleonenkoordinaten mitgenommen werden, verlangt, daß von K Glieder beliebig hoher Ordnung in g_0 berücksichtigt werden müssen ([16]).

Ein eigenartiges Gegenstück zur « Doppelgraphengleichung » (5) ist die aus der mehrzeitigen renormierten Bethe-Salpetergleichung gewonnene Integralgleichung

$$(6)\qquad \varphi_t^r(|\boldsymbol{x}_1\boldsymbol{x}_2|) = \varphi_t^0(|\boldsymbol{x}_1\boldsymbol{x}_2|) + \int_{-\infty}^{+\infty} \mathrm{d}t' \int \mathrm{d}\boldsymbol{x}_1'\mathrm{d}\boldsymbol{x}_2' \, L(\boldsymbol{x}_1\boldsymbol{x}_2 t;\, \boldsymbol{x}_1'\boldsymbol{x}_2' t')\varphi_{t'}^r(|\boldsymbol{x}_1'\boldsymbol{x}_2'|)\,,$$

für die renormierte einzeitige Wellenfunktion $\varphi_t^r(|\boldsymbol{x}_1\boldsymbol{x}_2|) = Z_1^{-1}\varphi_t(|\boldsymbol{x}_1\boldsymbol{x}_2|)$ worin L als Entwicklung nach Potenzen der endlichen Kopplungskonstanten $g_1 = Z_1^{-1}Z_2 Z_3^{\frac{1}{2}} g_0$ lediglich durch die konvergenten Renormierungsfunktionen S_{F1}', A_{F1}', Γ_{31} und M_1 ausgedrückt ist. Gl. (6) kann in einem nächsten Schritt in eine Integralgleichung

$$(7)\qquad \varphi_t^{r++}(|\boldsymbol{x}_1\boldsymbol{x}_2|) = \varphi_t^0(|\boldsymbol{x}_1\boldsymbol{x}_2|) + \int_{-\infty}^{+\infty} \mathrm{d}t' \int \mathrm{d}\boldsymbol{x}_1' \, \mathrm{d}\boldsymbol{x}_2' \, N(\boldsymbol{x}_1\boldsymbol{x}_2 t;\, \boldsymbol{x}_1'\boldsymbol{x}_2' t')\varphi_{t'}^{r++}(|\boldsymbol{x}_1'\boldsymbol{x}_2'|)\,,$$

für die positiven Frequenzanteile $\varphi_t^{r++}(|\boldsymbol{x}_1\boldsymbol{x}_2|)$ umgewandelt werden ([5,8,17-19]). Ein besonderer Vorzug von Gl. (6) und (7) ist darin zu sehen, daß das Problem der Renormierung ganz klar abgetrennt ist. Beide Gleichungen enthalten keine unendlichen Renormierungskonstanten mehr, es genügt, bereits anderweitig errechnete Näherungen der konvergenten Renormierungsfunktionen einzusetzen.

Über die Güte der besprochenen einzeitigen Näherungsverfahren im Einzelnen kann vorläufig noch wenig gesagt werden. Sie scheinen alle, einschließlich der neuen Tamm-Dancoffmethode, auf Theorien schwacher Kopplung zugeschnitten und dürften wohl nur in besonderen Fällen auf starke Kopplungen anwendbar sein.

([16]) Vgl. die analogen Verhältnisse in der (alten) Tamm-Dancoffmethode, M. Lévy: *Phys. Rev.*, **90**, 72 (1952).

([17]) A. Klein: *Phys. Rev.*, **90**, 1101 (1953).

([18]) W. Macke: *Zeits. f. Naturf.*, **8a**, 599 u. 615 (1953).

([19]) K. Symanzik: *Nuovo Cimento*, **11**, 88 (1954).

Esperienze sull'effetto Čerenkov nell'aria.

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(ricevuto il 7 Aprile 1954)

Proseguendo nelle esperienze intese a rivelare l'effetto Čerenkov prodotto da singole particelle dei raggi cosmici negli aeriformi (1), si è costruita una nuova apparecchiatura allo scopo di rivelare i fotoni prodotti lungo 80 cm di percorso in miscele di aria e cloroformio, con indice di rifrazione dell'ordine di 1,01. Le prime prove dell'apparecchio sono state eseguite senza introdurvi vapori di cloroformio e si è constatato un effetto anche nell'aria a pressione atmosferica.

L'apparecchiatura (che verrà dettagliatamente descritta in un prossimo lavoro) è schematicamente rappresentata in fig. 1, e consta di un tubo a pareti interne speculari, limitato inferiormente da uno specchio parabolico, che concentra la luce sul catodo di un fotomoltiplicatore EMI 6260, disposto con l'asse orizzontale. L'insieme può essere ermeticamente chiuso a tenuta di gas, ed è completato da un telescopio di contatori: quelli del I gruppo (v. fig. 1) hanno area utile di 8×8 cm^2 e un'uguale area è individuata da quelli del II e III gruppo in coincidenza; il fotomoltiplicatore è disposto in modo che le particelle rivelate da tale telescopio non possono attraversarne il fotocatodo: si elimina così l'inconveniente della rivelazione diretta da parte di esso (v. nota (1)).

L'apparecchio, compresi i contatori, è girevole intorno a un asse orizzontale, di traccia *O*. Inoltre vi è uno schermo opaco *S*, mobile dall'esterno, che può essere interposto davanti al fotocatodo. Gli impulsi del fotomoltiplicatore passano attraverso un amplificatore con tempo di salita $2 \cdot 10^{-8}$ s e costante di tempo di discesa $5 \cdot 10^{-8}$ s, e, dopo essere stati discriminati e allungati, vengono fatti coincidere con quelli del telescopio di contatori.

Diamo qui i risultati di tre serie di misure, in ciascuna delle quali si è verificato che le fluttuazioni nelle singole prove risultavano normali.

La prima serie è stata ottenuta alternando varie prove, della durata complessiva di 94 ore, con l'apparecchio diritto (come in fig. 1) e capovolto, senza schermo davanti al fotocatodo. Si sono inviati alla coincidenza gli impulsi del F.M. (alimentato a 1 650 V) superiori all'incirca a 0,05 V. Le frequenze così ottenute sono:

Diritto senza schermo .	$2{,}7 \pm 0{,}25$/h,
Capovolto senza schermo	$1{,}04 \pm 0{,}143$/h,
Differenza	$1{,}66 \pm 0{,}29$/h.

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(1) A. Ascoli Balzanelli e R. Ascoli: *Nuovo Cimento*, **10**, 1345 (1953).

La seconda serie è stata ottenuta alternando varie prove della durata complessiva di 149 ore con l'apparecchio sempre diritto, senza schermo e con schermo davanti al fotocatodo. Gli impulsi del fotomoltiplicatore (alimentato ancora a circa 1 650 V) inviati alla coincidenza erano di altezza superiore a circa 0,09 V. Le frequenze ottenute sono:

Diritto senza schermo . 1.61 ±0,137/h,
Diritto con schermo . . 0,97 ±0,124/h,
Differenza . . . . . 0,64 ±0,185/h.

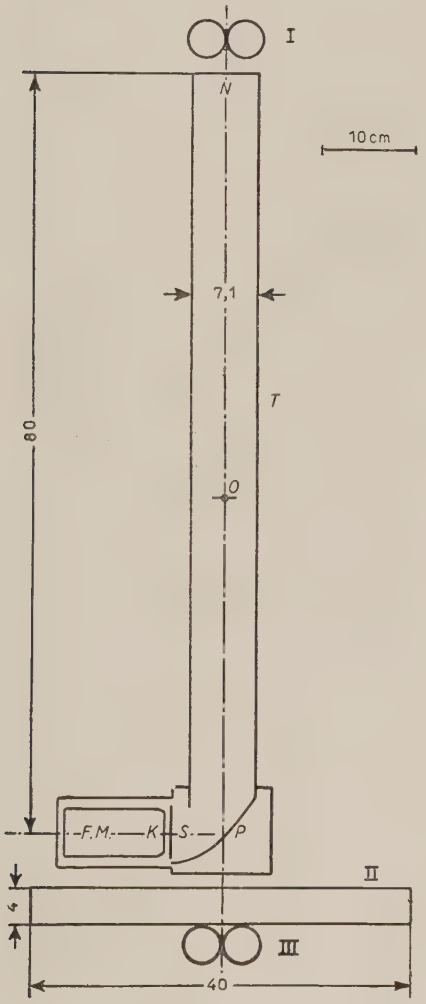

Fig. 1. - T, tubo a pareti speculari alluminato; P, specchio parabolico alluminato; S, schermo mobile dall'esterno; O, traccia dell'asse intorno a cui si può ruotare l'apparecchio; K, fotocatodo del fotomoltiplicatore F.M.; N, superficie interna annerita; I, II, III, telescopio di contatori; I; gruppo di due contatori lunghi 8 cm; II e III, gruppi formati ciascuno da due contatori lunghi 40 cm.

La terza serie è stata ottenuta dopo aver sostituito i due gruppi di contatori in coincidenza I e II con un solo gruppo, di ugual superficie utile (8×8 cm^2), formato da due contatori lunghi 8 cm. La tensione di alimentazione del F.M. è stata elevata a 1 800 V e si sono ancora inviati alla coincidenza gli impulsi di altezza superiore a 0,09 V. Si sono alternate alcune prove, della durata complessiva di 87 ore, con l'apparecchio diritto, senza e con schermo interposto, e capovolto senza schermo. Le frequenze ottenute sono:

a) Diritto senza schermo 1,56 ±0,20/h,
b) Diritto con schermo . 0,57 ±0,20/h,
c) Capovolto senza schermo . . . . . 0,765±0,15/h.

Differenze:

$$a) - b) = 0,99 \pm 0,28/\text{h},$$
$$a) - c) = 0,8 \pm 0,25/\text{h}.$$

Le differenze constatate tra le frequenze ottenute con apparecchio diritto, rispettivamente senza schermo e con schermo interposto, nella seconda e terza serie di misure, sono certamente attribuibili a fotoni prodotti dalle particelle dei raggi cosmici durante il loro percorso nell'aria all'interno dell'apparecchio. Questa produzione di fotoni si spiega come dovuta all'effetto Čerenkov prodotto dalle particelle» di velocità superiore a quella della luce nell'aria c/n($n = 1,000293$, indice di rifrazione dell'aria). Infatti il numero medio di fotoni emessi per unità di percorso da tali particelle, nell'intervallo di lunghezze d'onda tra 3 000 e 7 000 Å, calcolato con la formula di Frank e Tamm, varia da 0 a 0,51/cm, per velocità delle particelle comprese tra

c/n e c. Quindi in 80 cm di percorso possono venir prodotti al più 40 fotoni. Dato il rendimento del fotocatodo e del sistema ottico impiegato, si può ritenere che in media occorrano non più di 30 fotoni per produrre un fotoelettrone e vi sia quindi la possibilità di rivelare alcune particelle.

La frequenza degli eventi riscontrata è in accordo, come ordine di grandezza, con quella prevedibile in base all'intensità dei raggi cosmici: infatti il numero di particelle che percorrono interamente il tubo è di circa 15/h. Quelle di velocità sufficiente a produrre almeno 30 fotoni (mesoni di energia $> 9{,}3$ GeV, elettroni di energia > 43 MeV) si può ritenere che siano dell'ordine del 16%. Con tale calcolo approssimato si otterrebbe per tali particelle una frequenza di 2,2/h, dell'ordine di quella riscontrata sperimentalmente.

Il fatto che la produzione di fotoni constatata sia dovuta ad effetto Čerenkov nell'aria è confermato dalle misure della prima e terza serie effettuate con l'apparecchio capovolto: in tal caso infatti la luce prodotta per effetto Čerenkov viene assorbita dal fondo annerito dell'apparecchio (v. N in fig. 1).

Quanto agli impulsi registrati con lo schermo interposto ovvero con l'apparecchio capovolto, è escluso che siano dovuti a coincidenze casuali tra gli impulsi dati dal telescopio di contatori (~ 25/h) e quelli di fondo del F.M. (dell'ordine di 50/s), dato che queste risultano, secondo i calcoli, dell'ordine di 1/15 d. Si ritiene che tali impulsi siano invece dovuti a coincidenze provocate da sciami tali da colpire i contatori e in cui vi sia almeno una particella che attraversi il fotocatodo e venga rivelata direttamente da esso. Questa supposizione è confermata dalla terza serie di misure, che rivela una diminuzione nella frequenza di tali impulsi dopo la sostituzione dei due gruppi di contatori in coincidenza, lunghi 40 cm, con uno solo, formato da due contatori lunghi 8 cm, che ha reso il sistema dei contatori meno sensibile agli sciami.

La realizzazione dell'apparecchio è stata possibile grazie all'aiuto dell'ing. F. Carello, che ha provveduto nelle sue officine a risolvere il non facile problema della costruzione dello specchio parabolico e dell'alluminatura interna del medesimo e del tubo riflettente.

Ringraziamo inoltre il prof. G. Wataghin per aver seguito con vivo interessamento e costante appoggio la nostra ricerca.

Produzione di Mesoni π in Carbonio ed Alluminio.

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(ricevuto il 7 Aprile 1954)

La produzione di mesoni π da parte della radiazione cosmica in materiali condensati di diverso numero atomico (dall'alluminio al piombo) è stata studiata con la tecnica delle lastre nucleari da Dallaporta *et al.* [1] e da Bonetti *et al.* [2], con il risultato che le sezioni d'urto di produzione, dopo una probabile salita iniziale (da Al a S), sembrano decrescere con l'aumentare del numero atomico.

Allo scopo di completare tale ricerca ci siamo proposti col presente lavoro di studiare, con la stessa tecnica, la produzione di mesoni in materiali di numeri atomici inferiori a quello dello zolfo e precisamente, di confrontare la produzione in carbonio con quella in alluminio, onde avere una conferma della salita delle sezioni d'urto in tale intervallo ed un'indicazione sulla relativa legge di variazione.

Per tale studio abbiamo esaminato alcune lastre nucleari già opportunamente esposte alla radiazione cosmica sul monte Rosa (4 550 m s.l.m.) e precisamente: due lastre Ilford G5, di 1 200 μ di spessore, esposte verticalmente per 16 giorni, l'una contornata superiormente e lateralmente da uno spessore medio di 7 cm di grafite, l'altra da uno spessore medio di 5 cm di alluminio; si è esaminata inoltre una lastra dello stesso tipo, di 600 μ di spessore, esposta verticalmente, senza assorbitore, nello stesso luogo, dalla quale abbiamo dedotto i dati di fondo.

Dall'esplorazione delle lastre si sono ottenuti i valori riportati nelle colonne 2, 3, 4 della tabella I.

Nella colonna 1 sono indicati i volumi di emulsione esplorati (in cm^3) moltiplicati per il tempo di esposizione (in giorni). Il numero di mesoni π^- (colonna 5) è stato dedotto dal numero di mesoni σ osservati, tenendo conto che, come noto, l'1,5% dei mesoni μ producono stelle confondibili con stelle di σ e che il 27% dei mesoni π^- non producono disintegrazioni visibili e vengono quindi osservati come mesoni ρ. Il numero dei mesoni π^+ è stato preso uguale a quello dei decadimenti $\pi \to \mu$ osservati (colonna 4). Nella colonna 7 infine è riportato il numero di mesoni π per unità di volume dell'emulsione e per giorno di esposizione. Gli errori indicati accanto ai numeri della tabella sono quelli statistici.

Il numero T di mesoni π, prodotti per unità di tempo in ogni materiale e terminanti nell'unità di volume di emulsione delle lastre, è stato dedotto dal numero di mesoni riportato nella colonna 7

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(1) N. Dallaporta, M. Merlin, O. Pierucci e A. Rostagni: *Nuovo Cimento*, **9**, 202 (1952).

(2) A. Bonetti, N. Dallaporta, G. Dascola e M. Merlin: *Nuovo Cimento*, **10**, 215 (1953).

della tabella I sottraendo il numero di mesoni di fondo. Come fondo, supposto uguale nei due materiali, si è assunto — con le approssimazioni già indicate nel sopracitato lavoro [2] — il numero di mesoni ottenuto nelle lastre in aria (3ª riga della colonna 7).

nel volumetto dv_0 di emulsione, sono legati alla costante B, contenuta in $S(E)$, dalla relazione [1,2]:

$$(1)\quad T dv_0 = dv_0 \iiint e^{-\alpha z} \cdot e^{-R/\lambda} \cdot S(E) \frac{\tau(\theta)\sin\theta}{4\pi} \, dE \, d\theta \, d\varphi \, .$$

Tabella I.

Materiali	Vol. espl. × tempo ($cm^3 \cdot d$)	ϱ	σ	$\pi \to \mu$	π^-	$\pi^- + \pi^+$	$\frac{\pi^- + \pi^+}{cm^3 \cdot d}$
	1	2	3	4	5	6	7
Carbonio	55,2	830 ± 29	85 ± 9.2	98 ± 9,9	101 ± 13	199 ± 16	3,6 ± 0,29
Alluminio	59,1	775 ± 28	101 ± 10	76 ± 8,7	124 ± 14	200 ± 16	3,4 ± 0,27
Aria	127,2	1 751 ± 42	155 ± 12,4	123 ± 11,1	179 ± 17	302 ± 20	2,4 ± 0,16

Per quanto riguarda le correzioni da apportarsi a tali dati per le possibili deviazioni angolari dei mesoni dovute a scattering coulombiani singoli, nonchè per la produzione secondaria di mesoni dovuta a scattering anelastici dei mesoni stessi, la prima correzione è stata ritenuta trascurabile in base alle considerazioni già esposte nel predetto lavoro [2]; parimenti la seconda correzione non è stata eseguita, perchè sul rapporto dei numeri T, che interessa nei calcoli che seguono, essa importa una variazione che risulta piccola rispetto all'errore statistico.

I numeri T di mesoni prodotti nei materiali per giorno e terminanti nell'unità di volume (cm^3) dell'emulsione delle lastre risultano quindi:

$$T_{Al} = 1{,}0 \pm 0{,}3 \, ; \qquad T_C = 1{,}2 \pm 0{,}3 .$$

Assumendo d'altra parte per lo spettro di produzione dei mesoni nei materiali un'espressione della forma [1,2]:

$$S(E) = \text{cost.} = B \quad \text{per } E < E_l \cong 100 \text{ MeV},$$
$$S(E) = 1000 \cdot B/E^{\frac{8}{3}} \quad \text{per } E > E_l ,$$

si ha che i numeri $T dv_0$ di mesoni, prodotti per unità di tempo e terminanti dove α è il coefficiente di assorbimento della primaria, λ il libero cammino medio dei mesoni, R, θ e φ le coordinate polari nel sistema di riferimento scelto e $\tau(\theta)$ la funzione di distribuzione angolare dei mesoni prodotti, che si è assunta isotropa ($\tau(\theta) = 1$) per il range di energie che interessa nelle nostre condizioni sperimentali. In base alla relazione range-energia $E = kR^m$, $S(E)dE$ viene poi espressa in funzione di R (range dei mesoni).

I calcoli indicati nella relazione (1) sono stati eseguiti scegliendo, con sufficiente approssimazione, il volumetto dv_0 nel punto centrale delle lastre ed assumendo per i parametri m, k, α, λ, relativi ai due materiali impiegati, i valori riportati nella tabella II.

Si è ottenuto in definitiva:

$$T_{Al} = 42{,}5 \cdot B_{Al} \, ,$$
$$T_C = 42{,}5 \cdot B_C \, .$$

Il rapporto dei coefficienti di produzione, proporzionali ai fattori B, risulta allora nel nostro caso proprio uguale al rapporto T_{Al}/T_C, cioè:

$$\frac{B_{Al}}{B_C} = \frac{T_{Al}}{T_C} = \frac{1}{1{,}2} = 0{,}83 \pm 0{,}32 \, .$$

Per il rapporto delle sezioni d'urto di produzione nei due materiali — essendo $N_{Al} = 0{,}6 \cdot 10^{23}$ ed $N_C = 1{,}12 \cdot 10^{23}$ i numeri di nuclei per cm³ per l'alluminio e per il carbonio rispettivamente — si trova infine:

$$\frac{\sigma_{Al}}{\sigma_C} = \frac{B_{Al}}{B_C} \cdot \frac{N_C}{N_{Al}} = 1{,}6 \pm 0{,}6 .$$

Un valore del rapporto σ_{Al}/σ_C ($1{,}9 \pm \pm 0{,}2$) di poco superiore al precedente si sarebbe ottenuto se, approfittando della circostanza che le produzioni in alluminio ed in carbonio nel nostro caso sono risultate praticamente uguali (colonna 7 della tabella I), si fosse evitata la correzione del fondo (supposto ancora uguale nei due materiali), correzione che d'altra parte costituisce, come già segnalato (²), un punto alquanto discutibile di questo tipo di esperienze.

Poichè il rapporto delle sezioni d'urto geometriche nei due materiali impiegati è 1,7, i risultati precedenti indicherebbero che per materiali di basso numero atomico la sezione d'urto di produzione di mesoni cresce con il numero di massa A approssimativamente secondo una legge in $A^{\frac{2}{3}}$; e quindi con la stessa legge trovata nelle esperienze di fotoproduzione di mesoni e di produzione con protoni accelerati artificialmente.

Tabella II.

	m	k	α (cm²/g)	λ (g/cm²)
C	0,6	12,6	1/150	42
Al	0,6	11,3	1/200	55

Ringraziamo il prof. N. Dallaporta dell'Istituto di Fisica dell'Università di Padova per averci gentilmente fornito le lastre e per le utili discussioni ed il prof. G. Todesco, Direttore di questo Istituto, per l'interesse col quale ha seguito questo lavoro.

A Theory of the Electron.

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(ricevuto l'8 Aprile 1954)

In a recent paper (1) a theory of the electron was proposed based upon an analogy with the gravitational theory of matter. In this note a further development is made. The notation here is the same as that of the original paper to which reference may be made for details.

The analogy consists in the adoption of the following equation as a generalisation of Einstein's law of gravitation:

$$(1) \qquad P^{\mu\nu} - \tfrac{1}{2}\gamma^{\mu\nu}P = \chi T^{\mu\nu} .$$

$P^{\mu\nu}$ is a component of the curvature tensor in Riemannian space of five dimensions, P the curvature scalar and $T^{\mu\nu}$ is a component of the tensor which replaces the mass tensor of Einstein's theory. In the case in which μ, ν are placed equal to m, n with the values (1, 2, 3, 4), equation (1) can be written in the form (2):

$$(2) \qquad R^{mn} - \tfrac{1}{2}\, g^{mn}R = \\ = \chi(M^{mn} + S^{mn} - \Theta^{mn}) ,$$

R^{mn} being a component of the curvature tensor in four dimensions and R the curvature scalar.

As in the original paper M^{mn} denotes a component of the electromagnetic energy tensor and in this theory is geometrical in origin, its factor χ having the value $\gamma_{55}\alpha^2/2$.

A simple form of mass tensor in the theory of relativity is obtained by placing $\Theta^{mn} = \mu u^m u^n$, where μ denotes the density of matter and u^m, u^n denote velocity components. The expression (30) of the paper is the quantum mechanical form of the tensor.

In the theory a vector (θ^m) and a tensor (θ^{mn}) were introduced in addition to the electromagnetic potential (φ^m) and field components (B^{mn}). The terms (S^{mn}) are formed from products of the θ-components and those of the electromagnetic field and have the form of interaction terms.

From applications in the quantum theory, it appears that $\gamma_{55}\alpha^2 = e^2/m_0^2c^4$ and, moreover, this relation is necessary if the motion of an electron in an electromagnetic field is to be along a null geodesic (3). If, however, the constant χ is to be the same for fields containing charged or uncharged matter its value should be $-8\pi\varkappa/c^4$, where $\varkappa$ is the Newtonian constant of gravitation. The

(1) H. T. Flint and E. M. Williamson: *Nuovo Cimento*, **11**, 188 (1954).

(2) L. Rosenfeld: *Bull. Acad. Roy. de Belgique*, **13**, 6, 447 (1927).

(3) J. W. Fisher: *Proc. Roy. Soc.*, A **123**, 489 (1929).

identification of $-8\pi\varkappa/c^4$ with $e^2/2m_0^2c^4$ is clearly impossible for apart from the difference of sign these quantities are of different orders of magnitude. O. KLEIN [4] has pointed out that it appears necessary to attribute widely differing values to γ_{55} in different circumstances. It may be that χ has different values in the two cases or that it is necessary to write

$$\gamma_{55}\alpha^2/2 = -8\pi\varkappa/c^4 + \delta e^2/2m_0^2c^4 , \tag{3}$$

where $\delta=0$ where no charges are present but is unity where they exist. The addition of the first term on the right of equation (3) when $\delta=1$, has no appreciable effect, since the second term is overwhelming, but in the absence of charge the first term is important. In this case the curvature scalar is

$$R = 8\pi\varkappa T/c^4 ,$$

where T is the scalar $g_{mn}T^{mn}$ derived from the matter tensor (T^{mn}).

In the case where an electron is present the curvature scalar can be derived from equation (2) by making use of the value of $(M^{mn}+S^{mn})$ given in the paper and, remembering that $g_{mn}\mathfrak{S}^{mn}$ vanishes as a consequence of Dirac's equation, it follows that

$$R = \chi k^2 h^r h_r . \tag{4}$$

In order to examine the magnitude of the curvature let the static case be considered. In this case the component h^4 is the only one that exists and is equal to [5] $ie\exp[-kr]/r$ so that

$$R = -e^4k^2\exp[-2kr]/2m_0^2c^4r^2.$$

The value of k in this case is $2m_0c^2/e^2$ and if e^2/m_0c^2 be placed equal to r_0,

$$R = -2\exp[-4r/r_0]/r^2 ,$$

r denoting the distance to a point from the centre of the electron.

It is clear that the curvature falls off rapidly with distance. At the distance $r_0/2$ the curvature is of the order of $1/r_0^2$ but becomes rapidly less than the value 10^{-50} cm^{-2} which is the value adopted in Einstein's theory.

The geometrical picture suggested by these considerations is that of an extensive space of small curvature with minute regions superposed upon it of very great curvature. It may be supposed that within these intensely curved regions it is impossible to make measurements by the methods appropriate to the space outside them and that they are out of reach as are the regions outside the confines of a closed universe. In a geometrical theory of the electron of this kind the electromagnetic field is described in geometrical terms and is the expression of the geometry of space. The question why the charged particle is not disrupted by the action of Coulomb forces does not arise.

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[4] O. KLEIN: *Ark. f. Mat. Astr. o. Fys.*, **34**. H. 1 (1947).

[5] W. PAULI: *Handb. d. Phys.*, 2nd Edn., **24**/1, p. 235 (1933)

Assorbimento di ultrasuoni in miscele di elio-argon.

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(ricevuto il 10 Aprile 1954)

È ben noto che l'assorbimento del suono nei gas monoatomici puri segue le leggi classiche, stabilite da G. G. Stokes e G. Kirchhoff: l'assorbimento è in essi determinato dalla viscosità e dalla conduzione termica. Nel caso di miscugli di gas monoatomici, oltre le anzidette cause dissipatrici, intervengono, a determinare l'assorbimento, altri fattori, come la diffusione e la termodiffusione; la loro influenza risulta tanto più importante quanto più discosti sono i pesi molecolari dei componenti la miscela. Ricerche teoriche sull'assorbimento del suono nei miscugli di gas sono state fatte da tempo da Y. Rocard [2] e da M. Kohler [3]. Quest'ultimo partendo dalle equazioni della teoria cinetica dei gas, nella forma data loro da Enskog, ottiene per il coefficiente d'assorbimento α dell'ampiezza sonora l'espressione:

$$(1) \qquad \alpha = \frac{2\pi^2 f^2}{V^3}\left\{\frac{4}{3}\frac{\eta}{\varrho} + \frac{\gamma - 1}{\gamma}\frac{\lambda}{C_v\varrho} + \frac{\gamma - 1}{\gamma}\frac{\nu'_{12}\nu_{12}}{C_v\varrho} - \gamma D_{12}C_1C_2\left(\frac{M_2 - M_1}{M_0}\right)^2 + \right.$$
$$\left. + 2(\gamma - 1)\nu'_{12}C_1C_2\frac{M_2 - M_1}{M_0}\right\}.$$

In questa relazione i primi due termini sono quelli della formula di Kirchhoff, gli altri tre si riferiscono alla diffusione, alla termodiffusione e alla conduzione termica alterata per effetto della termodiffusione. Allo scopo di verificare questa teoria ho intrapreso delle misure di assorbimento di ultrasuoni in miscugli di gas e qui dò i primi risultati, ottenuti per la miscela elio-argon.

Viene per ciò adoperato un interferometro, tenuto a temperatura di 25 °C, nel quale la sorgente ultrasonora è costituita da una piastrina di quarzo vibrante su 3000 kHz. Il buon funzionamento dell'apparecchiatura è stato controllato misurando l'assorbimento nell'argon puro. Questo era ottenuto da una bombola del commercio e purificato facendolo passare in un forno contenente calcio metallico a 500 °C. Nella fig. 1 sono riportati i valori sperimentali di α/f^2 per tale gas, in funzione del rapporto f/p (f frequenza in Hz e p pressione in atmosfere); essi

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[1] L. Bergmann: *Der Ultraschall* (Stuttgart, 1949).

[2] Y. Rocard: *Journ. de Phys. et le Rad.*, **7**, 426 (1930).

[3] M. K. Kohler: *Ann. der Phys.*, **39**, 209 (1941).

si discostano poco dal valore classico ($190 \cdot 10^{-15}$ cm^{-1} s^2), anzi attorno alla frequenza di 10 MHz/atmosfere, che è quella, in vicinanza della quale sono state eseguite le misure sulle miscele, l'accordo

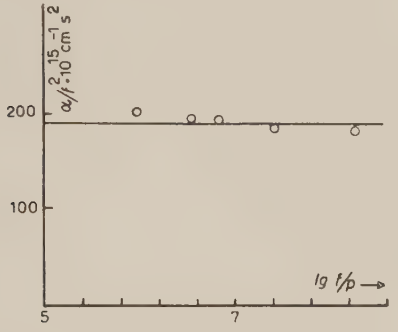

Fig. 1.

tra valori teorici e sperimentali si può ritenere molto buono. Anche il valore sperimentale di α/f^2 per l'elio risulta d'accordo con quello classico ($52 \cdot 10^{-15}$ cm^{-1}s^2)

Le misure sulle miscele di diversa concentrazione sono state fatte a pressioni, che da caso a caso potevano variare tra 15 e 40 cm Hg. Le concentrazioni sono state determinate in base alle pressioni parziali dei componenti; esse risultavano concordi con le concentrazioni che si potevano dedurre da un confronto tra i valori sperimentali della velocità di propagazione degli ultrasuoni nel miscuglio e quelli deducibili per tale grandezza dalla formula di Dixon e Greenwood (4):

$$(2) \qquad V^2 = \frac{RT}{M_1C_1 + M_2C_2} \cdot \frac{C_1C_{1p} + C_2C_{2p}}{C_1C_{1v} + C_2C_{2v}} .$$

La fig. 2 raccoglie i valori di α/f^2 calcolati in base alle determinazioni sperimentali del coefficiente di assorbimento dell'energia sonora per lunghezza d'onda, previa riduzione di questo a pressione atmosferica. L'assorbimento degli ultrasuoni nella miscela, in funzione della concentrazione dei componenti, presenta un massimo quando tali concentrazioni sono eguali. Ciò è d'accordo con la (1). I valori sperimentali risultano però superiori a quelli deducibili dalla (1). Così per $C_1 = C_2$, dall'esperienza risulta $\alpha/f^2 = 377 \cdot 10^{-15}$ cm^{-1} s^2, mentre il valore teorico è $252 \cdot 10^{-15}$ cm^{-1} s^2. Il calcolo è stato fatto modificando la (1) nell'ipotesi che le molecole si possano pensare come sfere elastiche rigide.

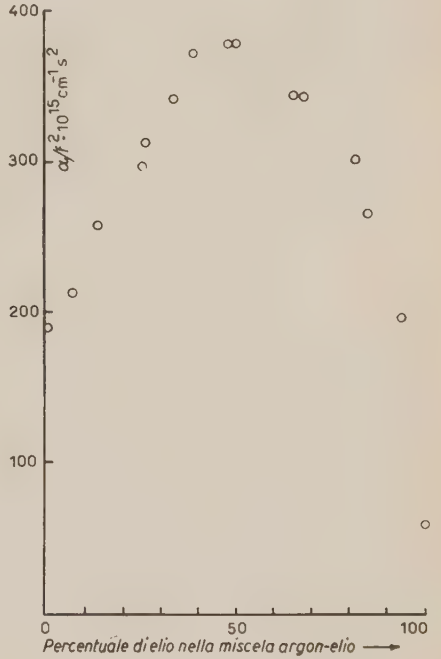

Fig. 2.

Questa discrepanza, che è molto superiore agli errori di misura, probabilmente può imputarsi alle approssimazioni fatte nello schema teorico o può anche interessare la legge di interazione tra molecole dissimili.

Le misure vengono proseguite su altre coppie di gas.

L'autore ringrazia il C.N.R., col cui contributo finanziario egli ha potuto intraprendere queste ricerche.

(4) H. B. Dixon e G. Greenwood: *Proc. Roy. Soc. London*, A, **109**, 561 (1925).

On the Production of Penetrating Showers in Hydrogen and Carbon.

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(ricevuto il 14 Aprile 1954)

The experimental arrangement used at altitude 3 500 m, in a series of earlier measurements [1] was improved recently by addition of two double hodoscopes of 30 Geiger counters in each in the manner indicated in Fig. 1.

The groups of counters 1, 2, 3, 4 inside the lead block register coincidences and serve as master for the hodoscopes. The measurements referred to here were made simultaneously on both arrangements with a layer of 12 g/cm² of graphite on one apparatus and a layer of 14 g/cm² of paraffin. The actual series of measurements has confirmed all the experimental results of the previous measurements [1]. The use of hodoscopes has permitted us to classify the observed events, taking into account the nature and the number of the incident particles:

1) Events due to dense penetrating showers (*DS*); these are registered when at least 11 counters of the tray *A* and the tray *B* are fired.

2) Events generated by one incident charged particle registered in the case when a number $N_A = 1$ of counters *A* is fired and a number $N_B \geqslant 3$ of the counters *B*. These « multiplication-showers » which can be due to protons, α-particles and π or K mesons, will be indicated as $MS_{(p)}$, because we think that in most cases the primary particle is a proton.

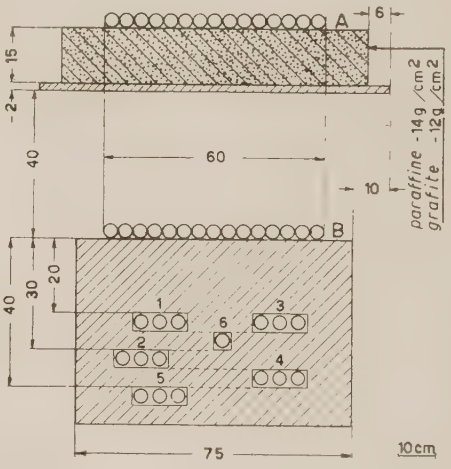

Fig. 1.

3) Events generated by neutral particles, which we suppose to be neutrons: they correspond to $N_A = 0$ counters of the tray *A* discharged and $N_B \geqslant 3$ of the tray *B* discharged.

[1] G. Bertolino, M. Cini, P. Colombino and G. Wataghin: *Nuovo Cimento*, **9**, 407 (1952).

The interpretation of the results will be made tentatively assuming that the *MS* showers 2) and 3) are due to the nucleons having energies $\geqslant 10$ GeV and the number of incident p and n is nearly the same. Indeed the production of *MS* 2) and 3) in graphite appears to be the same (s. Table I). Other events will be discussed in a more detailed paper.

The use of simultaneous measurements on two arrangements allows us to exclude the barometric effect, which cannot alter the observed differences

TABLE I.

	Paraffin (P)		Graphite (G)		Background (O)	$P-G$
	N	frequency	N	frequency	frequency	frequency
DS	686	0.51 ± 0.020	712	0.51 ± 0.02	0.53 ± 0.03	—
$MS_{(p)}$	638	0.461 ± 0.018	609	0.441 ± 0.018	0.19 ± 0.015	0.020 ± 0.025
$MS_{(n)}$	889	0.64 ± 0.022	670	0.48 ± 0.019	0.19 ± 0.015	0.160 ± 0.028

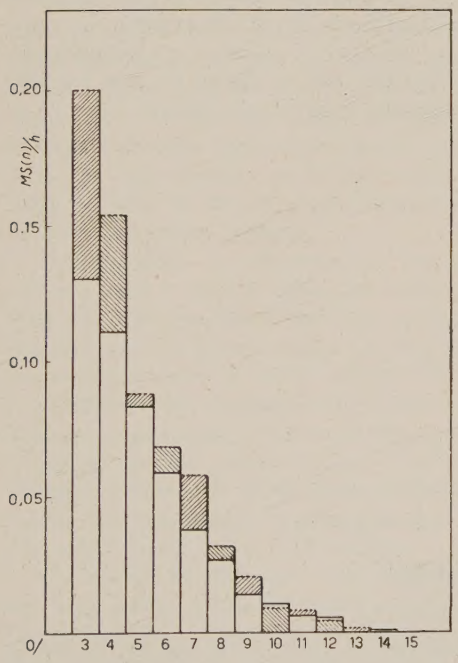

Fig. 2*a*. – Frequency per hour of «multiplication-showers» versus the number of registered charged particles due to a neutral incident particle. The difference $P - G$ is marked on the top of the block diagram.

	$P-O$	$G-O$	$P-G$
DS	—	—	—
$MS_{(p)}$	0.27	0.25	0.02
$MS_{(n)}$	0.45	0.29	0.16

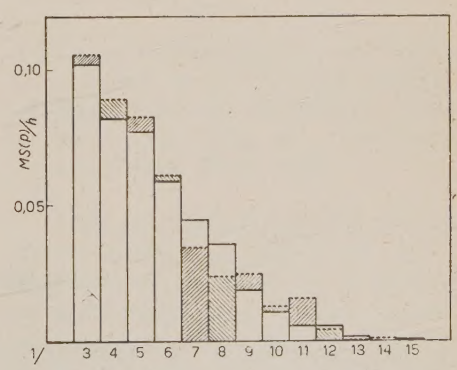

Fig. 2*b*. – Frequency per hour of «multiplication-showers» versus the number of registered charged particles due to a charged incident particle. The difference $P - G$ is marked on the top of the block diagram.

between the frequencies of events in paraffin and graphite.

The results of 1382 hours of measurements are given in Table I and in Fig. 2*a* and 2*b*. The production in hydrogen ($P - G$) of the MS with 3, 4 or 5 charged particles appears to have a greater frequency (cross-section) in the case of the incident $MS_{(n)}$ than in the case of incident proton $MS_{(p)}$. Indeed the ratio of these two frequencies appears to be $\geqslant 3$ (we cannot say more owing to the insufficient statistical data).

In the cases of higher multiplicity no difference between frequencies of the events due to charged and to neutral particles is observed. Further measurements are in progress.

The existence of a multiple production in the neutron-proton collision seems to follow from these measurements and confirms other recent results on multiple production. The effect of plural production is in the present experiment negligible (since a second collision with a carbon nucleus in paraffin can occur on the average in only 14% of the cases, if the interaction lenght l in paraffin is assumed to be $\sim 97\,g/cm^2$, and $\sim 20\%$, if $l \sim 60\ g/cm^2$).

On the basis of present results the statement made in a previous paper [1] on the average multiplicity ~ 10 in collisions with protons in paraffin must be corrected in a sense of a lower multiplicity ~ 3. The previous statement was due to the application of formulae which did not take into account multiplication events (production of secondary cascades and bursts) in the mass of the shield of Pb.

The asymmetry in the behaviour of neutrons and protons in the processes considered above can be due to the unknown mechanism of multiple production of pions and K-mesons.

We are grateful for cooperation in the preparation and execution of these experiments to: P. Brovetto. G. Cini, L. Gonella, G. Ghigo, A. Ferrero, F. Ferrero and C. Tribuno and for discussions to Prof. M. Verde.

ADDENDUM

A. Debenedetti, C. M. Garelli, G. Lovera, L. Tallone and M. Vigone: **An Analysis of Two Positive τ-Mesons**, *Nuovo Cimento*, **11**, 420 (1954).

We wish to point out that in our work we have taken into account only the errors which depend on the apparatus.

LIBRI RICEVUTI E RECENSIONI

P. M. Woodword B. A. – *Probability and information theory with application to radar*. London. 128 pag. con 20 fig. Pergamon Press, 1953.

Il libro fa parte della nuova serie di monografie *Electronics and Waves* diretta da D. W. Fry (Harwell).

La monografia è un vero strumento di lavoro: la lettura delle sue sole 120 pagine permette un completo inserimento nella letteratura odierna sull'argomento e può fornire lo spunto per ulteriori sviluppi: ciò è d'altronde nelle intenzioni del direttore della collana che intende fornire allo « specialista », in forma condensata, le basi di lavoro negli argomenti trattati dai singoli volumi.

L'esposizione è stringatissima, senza che ciò vada a scapito della chiarezza, e non c'è frase nel testo che non venga successivamente ripresa, tanta è la stretta connessione logica di tutta l'esposizione.

Il lettore, che si presuppone dotato di una certa maturità, viene condotto con completa coerenza, senza bisogno di riferimenti ad altri testi, dai fondamenti della teoria delle probabilità ai principali risultati della teoria della informazione.

Le teorie di Shannon sono esposte con originalità, evidente frutto di meditata rielaborazione. Ben scelti esempi facilitano l'assimilazione dei concetti introdotti e i relativi risultati numerici aiutano a concretamente usare grandezze fisiche che, come la « quantità di informazione », erano sfuggite, fino a pochi anni fa, a una precisa definizione operativa che permettesse la loro misura.

Le applicazioni al radar, campo in cui l'autore è particolarmente competente e in cui ha dato contributi originali, appagano la legittima curiosità del lettore, nuovo alla teoria dell'informazione, di veder applicata a un caso pratico una teoria così generale. È questa generalità della teoria, nata nel campo delle comunicazioni elettriche, ma applicabile anche al passaggio di informazione attraverso gli strumenti di misura impiegati in ogni esperienza di fisica sperimentale, che rende attraente e potenzialmente utile la lettura a una larga cerchia di studiosi.

L'edizione è curata e non sono stati rilevati errori.

E. Gatti

Colloques Internationaux du Centre National de la Recherche Scientifique. XXXIX. *Électrolyse*. Paris, 23-27 Mai 1952, IV + 147 pagg. ed. CNRS, Paris 1952.

Nella serie dei Colloqui Internazionali del Centro Nazionale della Ricerca Scientifica francese, il volume 39° è dedicato ai lavori riportati nel colloquio internazionale sull'elettrolisi, tenuto a Parigi dal 23 al 27 maggio del 1952. Quantunque il colloquio avesse come oggetto principale lo studio dei sali fusi e dell'interazione solvente-soluto, non sono mancati lavori su argomenti affini, e cioè su alcuni

aspetti termodinamici, quale ad esempio quello di PRIGOGINE sulla termodinamica della materia in un campo elettromagnetico, e quello sulla definizione termodinamica dei coefficienti di attività. Le relazioni sui sali fusi riguardavano principalmente metodi generali di studio e, tra questi, particolare menzione merita una esposizione sulla separazione isotopica per migrazione di ioni.

Sull'interazione fra soluto e solvente si sono avute numerose comunicazioni su argomenti molto diversi, tra i quali di notevole significato quelle relative alla diffusione della luce per opera delle soluzioni e la dimostrazione di effetti di combinazione delle soluzioni. La pubblicazione rappresenta un importantissimo documento di aggiornamento su aspetti fondamentali del fenomeno dell'elettrolisi, ed è quindi molto raccomandabile a tutti gli studiosi dell'argomento.

ANTONIO CARRELLI

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